1. Fix the vector spaces $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$. Consider the projection map $f\colon V \to W$, which is a homomorphism.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

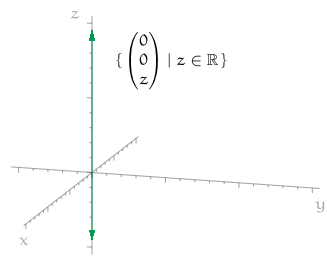
a) What members of the domain are mapped to this $\vec{w}$?

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solution: This is the inverse image of the zero vector.

$$f^{-1}(\vec{0}) = \{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \mid z \in \mathbb{R} \}$$

In general, the inverse image of $\vec{0} \in W$ is a subspace of V. It is the null space. Here is a drawing.



b) What members of the domain are mapped to this member of W?

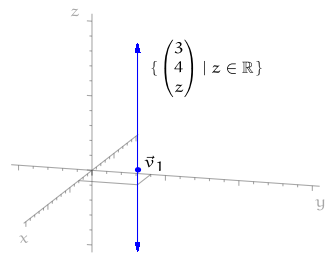
$$\vec{w}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Name one such vector, $\vec{v}_1$

Solution: This is the inverse image of $\vec{w}_1$.

$$f^{-1}(\begin{pmatrix} 3 \\ 4 \end{pmatrix}) = \{ \begin{pmatrix} 3 \\ 4 \\ z \end{pmatrix} \mid z \in \mathbb{R} \}$$

Here is a drawing.



One member of that set is this.

$$\vec{v_1} = \begin{pmatrix}3\\4\\1\end{pmatrix}$$

c) What members of the domain are mapped to this member of W?

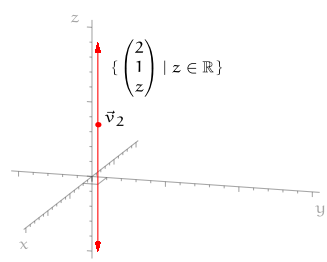
$$\vec{w}_2 = \begin{pmatrix}2\\1\end{pmatrix}$$

Produce one such vector, $\vec{v}_2$.

Solution: This is the inverse image of $\vec{w}_2$.

$$f^{-1}(\begin{pmatrix}2\\1\end{pmatrix}) = \{\begin{pmatrix}2\\1\\z\end{pmatrix} \mid z \in \mathbb{R}\}$$

Here is a drawing.



One member of that set is this.

$$\vec{v_2} = \begin{pmatrix}2\\1\\4\end{pmatrix}$$

d) In general, what members of the domain are mapped to this vector?

$$\vec{w} = \begin{pmatrix}a\\b\end{pmatrix}$$

Solution: This is the inverse image of $\vec{w}$.

$$f^{-1}(\vec{w}) = \{ \begin{pmatrix} a \\ b \\ z \end{pmatrix} \mid z \in \mathbb{R} \}$$

One member of that set is this.

$$\vec{v_2} = \begin{pmatrix} a \\ b \\ -3 \end{pmatrix}$$

e) Observe that $\vec{w}_1 + \vec{w}_2$ gives this.

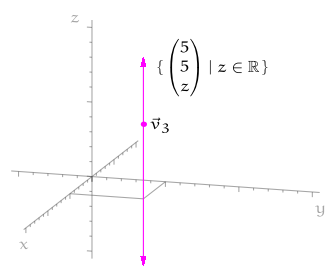
$$\vec{w}_3 = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

What members of the domain are mapped to this $\vec{w}_3$?

Solution: This is the inverse image of $\vec{w}_3$.

$$f^{-1}(\begin{pmatrix} 5 \\ 5 \end{pmatrix}) = \{ \begin{pmatrix} 5 \\ 5 \\ z \end{pmatrix} \mid z \in \mathbb{R} \}$$

Here is a drawing.



One member of that set is this.

$$\vec{v}_3 = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

f) Add $\vec{v}_1 + \vec{v}_2$. Observe that it is a member of the inverse image of $\vec{w}_3$.

Solution: A blue vector plus a red vector makes a purple vector.

$$\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} = \vec{v}_3$$

g) Observe that $6 \cdot \vec{w}_2$ gives this.

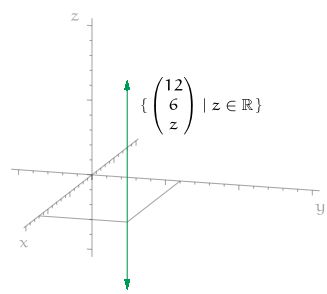
$$6\vec{w}_2 = \vec{w}_4 = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$$

What members of the domain are mapped to this $\vec{w}_4$?

Solution: This is the inverse image of $\vec{w}_4$.

$$f^{-1}(\begin{pmatrix} 12 \\ 6 \end{pmatrix}) = \{ \begin{pmatrix} 12 \\ 6 \\ z \end{pmatrix} \mid z \in \mathbb{R} \}$$

Here is a drawing.



One member of that set is this.

$$\vec{v}_6 = \begin{pmatrix} 12 \\ 6 \\ -1 \end{pmatrix}$$

h) Find $6 \cdot \vec{v}_2$. Observe that it is a member of the inverse image of $\vec{w}_6$.

Solution: We have

$$6\vec{v_2} = 6 \cdot \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \\ 24 \end{pmatrix}$$

and it projects to this vector.

$$f(6\vec{v_2}) = f(\begin{pmatrix} 12 \\ 6 \\ 24 \end{pmatrix}) = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$$

2. We can go through the same steps for this linear map $h\colon V \to W$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x+z \end{pmatrix}$$

a) What members of the domain are mapped to this $\vec{w}$?

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solution: This is the inverse image of the zero vector $\vec{0} \in W$.

$$h^{-1}(\vec{0}) = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y = 0 \text{ and } x + z = 0 \}$$

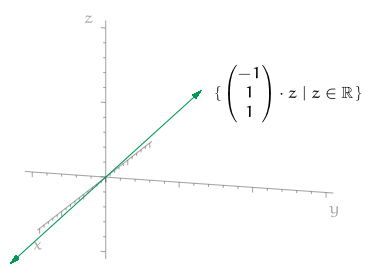
We get this linear system.

$$\begin{array}{rcl} x + y \quad = 0 & \xrightarrow{-\rho_1+\rho_2} & x + \quad y \quad = 0 \\ x \quad + z = 0 & & -y + z = 0 \end{array}$$

So we have that the null space is this.

$$\mathscr{N}(h) = \{ \begin{pmatrix} -z \\ z \\ z \end{pmatrix} \mid z \in \mathbb{R} \} = \{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot z \mid z \in \mathbb{R} \}$$

Here is the picture.



b) What members of the domain are mapped to this member of W?

$$\vec{w}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

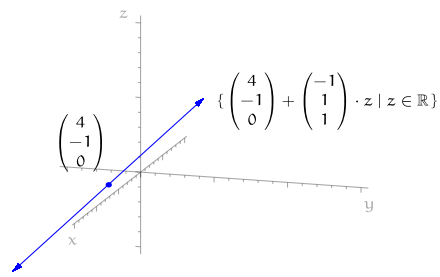
Solution: To find the inverse image of $\vec{w}_1$, we get this linear system.

$$\begin{array}{rcl} x + y \quad = 3 & \xrightarrow{-\rho_1+\rho_2} & x + \quad y \quad = 3 \\ x \quad + z = 4 & & -y + z = 1 \end{array}$$

So $y = -1 + z$ and $x = 4 - z$.

$$h^{-1}(\begin{pmatrix} 3 \\ 4 \end{pmatrix}) = \{ \begin{pmatrix} 4 - z \\ -1 + z \\ z \end{pmatrix} \mid z \in \mathbb{R} \} = \{ \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot z \mid z \in \mathbb{R} \}$$

Here is a drawing.



c) What members of the domain are mapped to this member of W?

$$\vec{w}_2 = \begin{pmatrix}2\\1\end{pmatrix}$$

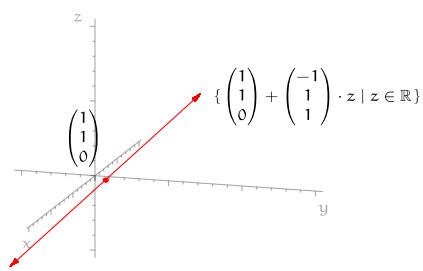
Solution: To find the inverse image of $\vec{w}_1$, we get this linear system.

$$\begin{array}{rcl} x+y \quad\quad &=& 2 \\ x \quad\quad +z &=& 1 \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rcl} x+\ \ y \quad\quad &=& 2 \\ -y+z &=& -1 \end{array}$$

So $y = 1+z$ and $x = 1-z$.

$$h^{-1}(\begin{pmatrix}2\\1\end{pmatrix}) = \{\begin{pmatrix}1-z\\1+z\\z\end{pmatrix} \mid z\in\mathbb{R}\} = \{\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}-1\\1\\1\end{pmatrix}\cdot z \mid z\in\mathbb{R}\}$$

Here is a drawing.



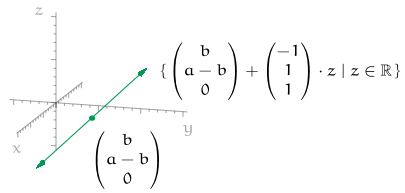
d) In general, what members of the domain are mapped to this vector?

$$\vec{w} = \begin{pmatrix}a\\b\end{pmatrix}$$

Solution: We get this linear system.

$$\begin{array}{rcl} x+y \quad\quad &=& a \\ x \quad\quad +z &=& b \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rcl} x+\ \ y \quad\quad &=& a \\ -y+z &=& -a+b \end{array}$$

So $y = (a - b) + z$ and $x = b - z$.

$$h^{-1}(\begin{pmatrix} a \\ b \end{pmatrix}) = \{ \begin{pmatrix} b - z \\ (a - b) + z \\ z \end{pmatrix} \mid z \in \mathbb{R} \} = \{ \begin{pmatrix} b \\ a - b \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot z \mid z \in \mathbb{R} \}$$

Here is a drawing.