When you are working on this material, you must work entirely on your own. You may use the book or your notes. But you may not work with other people, either in the class our outside of it, or use other books or the Internet. If you have any questions, email me.

1. Show that $h\colon \mathcal{P}_2 \to \mathbb{R}^3$

$$ax^2+bx+c \mapsto \begin{pmatrix} a-b \\ b \\ a+c \end{pmatrix}$$

is an isomorphism.

a) Verify that it is one-to-one.

Solution: Assume that $h(\vec{v}_1) = h(\vec{v}_2)$, so that $h(a_1x^2+b_1x+c_1) = h(a_2x^2+b_2x+c_2)$. Then

$$\begin{pmatrix} a_1-b_1 \\ b_1 \\ a_1+c_1 \end{pmatrix} = \begin{pmatrix} a_2-b_2 \\ b_2 \\ a_2+c_2 \end{pmatrix}$$

and the middle entries show that $b_1 = b_2$. With that, the top entries show that $a_1 = a_2$, and then the bottom entries give $c_1 = c_2$. The conclusion is that $\vec{v}_1 = \vec{v}_2$, and so h is a one-to-one function.

b) Verify that it is onto.

Solution: (*Hint:* try this first with numbers, that is, try two particular members of $\mathbb{R}^3$ to see the pattern for the general case.) Fix a member of the codomain $\mathbb{R}^3$.

$$\vec{w} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \in \mathbb{R}^3$$

We must find $\vec{v} \in \mathcal{P}_2$ so that $h(\vec{v}) = \vec{w}$. Consider ax^2+bx+c where $b = q$, where $a = p+q$, and where $c = r-(p+q)$. The verification that $\vec{v}$ is correct is this.

$$(p+q)x^2+qx+(r-(p+q)) \mapsto \begin{pmatrix} (p+q)-q \\ q \\ (p+q)+r-(p+q) \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

c) Verify that it preserves vector addition and scalar multiplication.

Solution: This shows that it preserves addition.

$$\begin{aligned}
h(\vec{v}_1+\vec{v}_2) &= h(\ (a_1x^2+b_1x+c_1)+(a_2x^2+b_2x+c_2)\ ) \\
&= h(\ (a_1+a_2)x^2+(b_1+b_2)x+(c_1+c_2)\ ) \\
&= \begin{pmatrix} (a_1+a_2)-(b_1+b_2) \\ b_1+b_2 \\ (a_1+a_2)+(c_1+c_2) \end{pmatrix} \\
&= \begin{pmatrix} (a_1-b_1)+(a_2-b_2) \\ b_1+b_2 \\ (a_1+c_1)+(a_2+c_2) \end{pmatrix} \\
&= \begin{pmatrix} a_1-b_1 \\ b_1 \\ a_1+c_1 \end{pmatrix} + \begin{pmatrix} a_2-b_2 \\ b_2 \\ a_2+c_2 \end{pmatrix} \\
&= h(a_1x^2+b_1x+c_1)+h(a_2x^2+b_2x+c_2) = h(\vec{v}_1)+h(\vec{v}_2)
\end{aligned}$$

And this is the similar calculation for preservation of scalar multiplication.

$$\begin{aligned}
h(r\cdot\vec{v}) &= h(\ r\cdot(ax^2+bx+c)\ ) \\
&= h(\ (ra)x^2+(rb)x+(rc)\ ) \\
&= \begin{pmatrix} ra-rb \\ rb \\ ra+rc \end{pmatrix} \\
&= r\cdot\begin{pmatrix} a-b \\ b \\ a+c \end{pmatrix} \\
&= r\cdot h(ax^2+bx+c) = r\cdot h(\vec{v})
\end{aligned}$$