1. Consider this matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

a) Find its characteristic polynomial.

Solution: The characteristic equation is

$$0 = \begin{vmatrix} 1-x & 1 & 1 \\ 0 & -x & 1 \\ 0 & 0 & 1-x \end{vmatrix} = (1-x)^2(-x)$$

b) Find its eigenvalues.

Solution: The eigenvalues are $\lambda_1 = 1$ (this is a repeated root of the equation) and $\lambda_2 = 0$.

c) Find the eigenvectors associated with the first eigenvalue.

Solution: Consider this system.

$$\begin{aligned} (1-x)\cdot b_1 + \quad b_2 + \quad b_3 &= 0 \\ -x\cdot b_2 + \quad b_3 &= 0 \\ (1-x)\cdot b_3 &= 0 \end{aligned}$$

For $\lambda_1 = 1$ we get this system.

$$\begin{aligned} 0\cdot b_1 + \quad b_2 + \quad b_3 &= 0 \\ -1\cdot b_2 + \quad b_3 &= 0 \\ 0\cdot b_3 &= 0 \end{aligned}$$

The sum of the first two equations gives $2b_3 = 0$, and so $b_3 = 0$ With that, back substitution gives $b_2 = 0$. The solution set is this eigenspace.

$$V_1 = \{ \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix} \mid b_1 \in \mathbb{C} \} = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot b_1 \mid b_1 \in \mathbb{C} \}$$

d) Find the eigenvectors associated with the second eigenvalue.

Solution: Again consider this system.

$$\begin{aligned}(1-x)\cdot b_1 + \quad b_2 + \quad b_3 &= 0\\ -x\cdot b_2 + \quad b_3 &= 0\\ (1-x)\cdot b_3 &= 0\end{aligned}$$

When $\lambda_2 = 0$ we have this.

$$\begin{aligned}b_1 + b_2 + b_3 &= 0\\ b_3 &= 0\\ b_3 &= 0\end{aligned}$$

The solution set is this eigenspace.

$$V_0 = \{\begin{pmatrix}-b_2\\ b_2\\ 0\end{pmatrix} \mid b_2 \in \mathbb{C}\} = \{\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}\cdot b_2 \mid b_2 \in \mathbb{C}\}$$

2. Consider the differentiation operator $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_3$. From Calculus we know it is linear.

$$\frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g \qquad \frac{d}{dx}r\cdot f = r\cdot\frac{d}{dx}f$$

a) Represent this map with respect to the natural basis $B = \langle 1, x, x^2, x^3\rangle$.

Solution: The map's action is $1 \mapsto 0$, $x \mapsto 1$, $x^2 \mapsto 2x$, and $x^3 \mapsto 3x^2$. So its representation is easy to compute.

$$T = \mathrm{Rep}_{B,B}(d/dx) = \begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{pmatrix}_{B,B}$$

b) Find its characteristic equation.

Solution: The computation is easy because the matrix is already in echelon form.

$$0 = |T - xI| = \begin{vmatrix}-x&1&0&0\\0&-x&2&0\\0&0&-x&3\\0&0&0&-x\end{vmatrix} = x^4$$

c) List the eigenvalues.

Solution: The map has the single eigenvalue $\lambda = 0$ (it is a repeated root of the characteristic equation).

d) Find the associated eigenspaces.

Solution: Solve this matrix equation.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{B,B} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}_B = 0 \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}_B$$

We get this.

$$\begin{aligned} 0b_1 + 1 \cdot b_2 \qquad\qquad &= 0 \\ 2 \cdot b_3 \qquad &= 0 \\ 3 \cdot b_4 &= 0 \\ 0 &= 0 \end{aligned}$$

The conclusion is that $b_2 = b_3 = b_4 = 0$, and b_1 is free. This is the set of representatives.

$$\{ \begin{pmatrix} b_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_B \mid b_1 \in \mathbb{C} \}$$

And here is the eigenspace associated with the eigenvalue $\lambda = 1$.

$$V_0 = \{ b_1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \mid b_1 \in \mathbb{C} \} = \{ b_1 \mid b_1 \in \mathbb{C} \}$$

Of course, we recognize from Calculus that the collection of constant polynomials is indeed nulled out by the derivative operation.