1. Consider this subset of the collection of all quadratic polynomials, $\mathcal{P}_2$.

$$S = \{px^2 + qx + r \in \mathcal{P}_2 \mid p + 2q - r = 0\}$$

a) List two elements and two non-elements.

Solution: Here are two elements. They satisfy the requirement that the coefficient of x^2 plus twice the coefficient of x minus the constant term equals zero.

$$x^2 + 2x + 5 \qquad 2x^2 + 2$$

Here are two non-elements.

$$x^2 + 2x - 4 \qquad x + 2$$

b) Show that it is closed under addition and also closed under scalar multiplication.

Solution: Fix $\vec{v}, \vec{w} \in S$ and take

$$\vec{v} = v_2x^2 + v_1x + v_0 \qquad \vec{w} = w_2x^2 + w_1x + w_0$$

where $v_2 + 2v_1 - v_0 = 0$ and $w_2 + 2w_1 - w_0 = 0$. The sum of the two is

$$\vec{v} + \vec{w} = (v_2 + w_2) \cdot x^2 + (v_1 + w_1) \cdot x + (v_0 + w_0)$$

and $(v_2 + w_2) + 2(v_1 + w_1) - (v_0 + w_0) = (v_2 + 2v_1 - v_0) + (w_2 + 2w_1 - w_0) = 0$. Thus $\vec{v} + \vec{w} \in S$.

For closure under scalar multiplication, take $r \in \mathbb{R}$. This is the scalar multiple of the vector.

$$r\vec{v} = r(v_2x^2 + v_1x + v_0) = (rv_2) \cdot x^2 + (rv_1) \cdot x + (rv_0)$$

Because $rv_2 + 2rv_1 - rv_0 = r \cdot (v_2 + 2v_1 - v_0) = 0$, the set S is closed under scalar multiplication.

c) Show that it is closed under linear combinations of two elements.

Solution: Fix $\vec{v} = v_2x^2 + v_1x + v_0$ and $\vec{w} = w_2x^2 + w_1x + w_0$, where $v_2 + 2v_1 - v_0 = 0$ and $w_2 + 2w_1 - w_0 = 0$. Let $r, s \in \mathbb{R}$. Then $r\vec{v} + s\vec{w} = (rv_2)x^2 + (rv_1)x + (rv_0) + (sw_2)x^2 + (sw_1)x + (sw_0)$, and that equals $(rv_2 + sw_2)x^2 + (rv_1 + sw_1)x + (rv_0 + sw_0)$. Now, $(rv_2 + sw_2) + 2(rv_1 + sw_1) - (rv_0 + sw_0)$ equals $(rv_2 + 2rv_1 - rv_0) + (sw_2 + 2sw_1 - +sw_0) = 0$, so the linear combination is also a member of S.

2. Each of these is a subspace. Parametrize each to find a spanning set.

a) This is a subspace of $\mathcal{P}_2$.

$$S_1 = \{px^2 + qx + r \in \mathcal{P}_2 \mid p + 2q - r = 0\}$$

Solution: Consider $p + 2q - r = 0$ to be a linear system with one equation. Then writing the leading variables in terms of those that are free gives $p = -2q + r$. We have

$$\begin{aligned} S_1 &= \{(-2q + r) \cdot x^2 + q \cdot x + r \mid q, r \in \mathbb{R}\} \\ &= \{(-2x^2 + x) \cdot q + (x^2 + 1) \cdot r \mid q, r \in \mathbb{R}\} \end{aligned}$$

The spanning set has two elements, $\{-2x^2 + x, x^2 + 1\}$.

b) This is a subspace of $\mathcal{M}_{2\times 2}$.

$$S_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a - 2b = 0 \text{ and } 2c + d = 0\}$$

Solution: The restrictions make a linear system.

$$\begin{aligned} a - 2b \qquad\quad &= 0 \\ 2c + d &= 0 \end{aligned}$$

Writing its leading variables in terms of its free variables gives $a = 2b$ and $c = (-1/2)d$.

$$\begin{aligned} S_2 &= \{\begin{pmatrix} 2b & b \\ -d/2 & d \end{pmatrix} \mid b, d \in \mathbb{R}\} \\ &= \{\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \cdot b + \begin{pmatrix} 0 & 0 \\ -1/2 & 1 \end{pmatrix} \cdot d \mid b, d \in \mathbb{R}\} \end{aligned}$$

Here is a spanning set.

$$\{\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1/2 & 1 \end{pmatrix}\}$$

c) This is a subspace of $\mathbb{R}^3$.

$$S_3 = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid 2x - z = 0\}$$

Solution: Here also, consider the restriction $2x - z = 0$ to be a one-equation linear system, to get that $x = (1/2)z$. But don't miss that also free is y.

$$\begin{aligned} S_3 &= \{ \begin{pmatrix} z/2 \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R} \} \\ &= \{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot y + \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} \cdot z \mid y, z \in \mathbb{R} \} \end{aligned}$$

With that, a spanning set is this.

$$\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} \}$$

d) Another subspace of $\mathbb{R}^3$.

$$S_4 = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y - z = 0 \text{ and } 2x + 2y = 0 \}$$

Solution: Finally, a linear system that requires some of Gauss's method.

$$\begin{aligned} x + \ \ y - z &= 0 \\ 2x + 2y \qquad &= 0 \end{aligned} \quad \xrightarrow{-2\rho_1 + \rho_2} \quad \begin{aligned} x + y - \ \ z &= 0 \\ 2z &= 0 \end{aligned}$$

Conclusion: leading are x and z, while y is free. The restriction is that $x = -y$ and $z = 0$.

$$S_4 = \{ \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} \mid y \in \mathbb{R} \} = \{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot y \mid y \in \mathbb{R} \}$$

This set is a line through the origin. This is a spanning set.

$$\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \}$$