1. Consider this subset of $\mathbb{R}^2$.

$$B = \langle \vec{\beta}_1, \vec{\beta}_2 \rangle = \langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \rangle$$

a) We will show that it is a basis. First, show that it is linearly independent.

Solution: For linear independence,

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

gives a linear system

$$\begin{aligned} 2c_1 + \ \ c_2 &= 0 \\ c_1 + 3c_2 &= 0 \end{aligned}$$

with this solution.

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & 3 & 0 \end{array}\right) \xrightarrow{(-1/2)\rho_1+\rho_2} \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 5/2 & 0 \end{array}\right)$$

So there is only one solution, the trivial one that $c_1 = c_2 = 0$.

b) Finish showing that the set B is a basis by showing that it spans the vector space $\mathbb{R}^2$.

Solution: To show that the set B spans the space $\mathbb{R}^2$, take a member of that space and show we can express it as a combination of vectors from B.

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

gives a linear system

$$\begin{aligned} 2c_1 + \ \ c_2 &= x \\ c_1 + 3c_2 &= y \end{aligned}$$

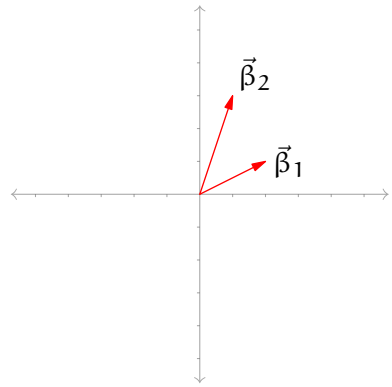
with this solution.

$$\left(\begin{array}{cc|c} 2 & 1 & x \\ 1 & 3 & y \end{array}\right) \xrightarrow{(-1/2)\rho_1+\rho_2} \left(\begin{array}{cc|c} 2 & 1 & x \\ 0 & 5/2 & -(1/2)x+y \end{array}\right)$$

So there is a solution: given an x and y, we can find c_1 and c_2. Specifically, $c_2 = -(1/5)x + (2/5)y$ and $c_1 = (3/5)x - (1/5)y$.

c) Plot the two vectors $\vec{\beta}_1$ and $\vec{\beta}_2$ on the xy-plane.

Solution:



d) In place of B, consider this set.

$$D = \langle \vec{\delta}_1, \vec{\delta}_2 \rangle = \langle \vec{\beta}_1, \vec{\beta}_1 + (1/10)\vec{\beta}_2 \rangle = \langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2.1 \\ 1.3 \end{pmatrix} \rangle$$

Show that it is linearly independent.

Solution: The setup

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 21/10 \\ 13/10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

leaves a system that is a little awkward by hand

$$\left(\begin{array}{cc|c} 2 & 21/10 & 0 \\ 1 & 13/10 & 0 \end{array}\right) \xrightarrow{(-1/2)\rho_1+\rho_2} \left(\begin{array}{cc|c} 2 & 21/10 & 0 \\ 0 & 1/4 & 0 \end{array}\right)$$

but again we have a single solution, the trivial one that $c_1 = 0$ and $c_2 = 0$.

e) Show that the set D spans the space $\mathbb{R}^2$.

Solution: The setup

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 21/10 \\ 13/10 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

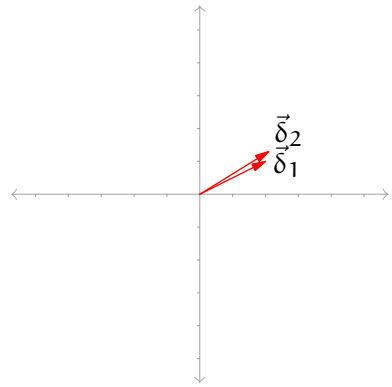
gives this.

$$\left(\begin{array}{cc|c} 2 & 21/10 & x \\ 1 & 13/10 & y \end{array}\right) \xrightarrow{(-1/2)\rho_1+\rho_2} \left(\begin{array}{cc|c} 2 & 21/10 & x \\ 0 & 1/4 & -(1/2)x+y \end{array}\right)$$

Again, there is a solution. That is, given x and y, there are c_1 and c_2 to make this work. So D spans the space.

f) Plot the members of D.

Solution: There is in the new vector $\vec{\delta}_2$ just a little bit of the second vector in the basis B. But even that little bit is enough to keep the set linearly independent.



2. Consider this subspace of $\mathbb{R}^4$.

$$S = \{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \mid 2x - z + w = 0 \}$$

a) Find a basis.

Solution: We have

$$S = \{ \begin{pmatrix} (1/2)z - (1/2)w \\ y \\ z \\ w \end{pmatrix} \mid y, z, w \in \mathbb{R} \} = \{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot y + \begin{pmatrix} 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot z + \begin{pmatrix} -1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot w \mid y, z, w \in \mathbb{R} \}$$

and checking that this set

$$B = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

is linearly independent is routine.

b) Using the method of the prior question, take mixtures of basis vectors to generate three more bases.

Solution: Obviously a person can take any mixtures that they like. You just have to keep components of the three in there, so it will still span, and keep the set of three still independent. One mixture is this,

$$B_1 = \langle \vec{\beta}_1, \vec{\beta}_1 + \vec{\beta}_2, \vec{\beta}_1 + \vec{\beta}_3 \rangle = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \rangle$$

a second one is this,

$$B_2 = \langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_2 + \vec{\beta}_3 \rangle = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \rangle$$

and a third is this.

$$B_3 = \langle (1/3)\vec{\beta}_2, \vec{\beta}_2 + \vec{\beta_3}, 3\vec{\beta}_1 \rangle = \langle \begin{pmatrix} 1/6 \\ 0 \\ 1/3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix} \rangle$$