Consider this linear transformation $t\colon \mathbb{R}^2 \to \mathbb{R}^2$.

$$t(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 3x+2y \\ 3x+8y \end{pmatrix}$$

We will illustrate that the eigenvalues are a feature of the map.

1. Fix the basis

$$B = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$$

a) Represent the map t with respect to B, B.

Solution: This is the action of the map on the basis.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xmapsto{t} \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \xmapsto{t} \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

The representation of the first outcome with respect to B is this.

$$\begin{pmatrix} 3 \\ 3 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathrm{Rep}_B(\begin{pmatrix} 3 \\ 3 \end{pmatrix}) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

The second outcome gives this.

$$\begin{pmatrix} 5 \\ 11 \end{pmatrix} = -6 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 11 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathrm{Rep}_B(\begin{pmatrix} 5 \\ 11 \end{pmatrix}) = \begin{pmatrix} -6 \\ 11 \end{pmatrix}$$

The matrix is this.

$$\mathrm{Rep}_{B,B}(t) = \begin{pmatrix} 0 & -6 \\ 3 & 11 \end{pmatrix}$$

b) Use the matrix $\mathrm{Rep}_{B,B}(t)$ to compute the characteristic equation.

Solution: We find $|T - x \cdot I|$ as here.

$$0 = \begin{vmatrix} 0-x & -6 \\ 3 & 11-x \end{vmatrix} = (-x)(11-x) + 18 = x^2 - 11x + 18$$

c) Find the eigenvalues.

Solution: The roots are

$$0 = x^2 - 11x + 18 = (x-2)(x-9)$$

$\lambda_1 = 2$ and $\lambda_2 = 9$.

2. Fix this basis.

$$D = \langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \rangle$$

a) Represent the map t with respect to D, D.

Solution: This is the action of the map on the basis.

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \xmapsto{t} \begin{pmatrix} 8 \\ 14 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 3 \end{pmatrix} \xmapsto{t} \begin{pmatrix} 6 \\ 24 \end{pmatrix}$$

The representation of the first with respect to D is this.

$$\begin{pmatrix} 8 \\ 14 \end{pmatrix} = 4 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + (10/3) \cdot \begin{pmatrix} 0 \\ 3 \end{pmatrix} \qquad \mathrm{Rep}_D(\begin{pmatrix} 8 \\ 14 \end{pmatrix}) = \begin{pmatrix} 4 \\ 10/3 \end{pmatrix}$$

The second outcome gives this.

$$\begin{pmatrix} 6 \\ 24 \end{pmatrix} = 3 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 7 \cdot \begin{pmatrix} 0 \\ 3 \end{pmatrix} \qquad \mathrm{Rep}_D(\begin{pmatrix} 6 \\ 24 \end{pmatrix}) = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

The matrix is this.

$$\mathrm{Rep}_{D,D}(t) = \begin{pmatrix} 4 & 3 \\ 10/3 & 7 \end{pmatrix}$$

b) Use the matrix $\mathrm{Rep}_{D,D}(t)$ to compute the characteristic equation.

Solution: Here it is.

$$0 = |T - x \cdot I| = \begin{vmatrix} 4-x & 3 \\ 10/3 & 7-x \end{vmatrix} = (4-x)(7-x) - 10 = x^2 - 11x + 18$$

c) Find the eigenvalues.

Solution: The characteristic equation $0 = x^2 - 11x + 18$ is the same so the roots are the same: $\lambda_1 = 2$ and $\lambda_2 = 9$.