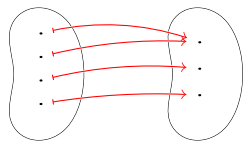
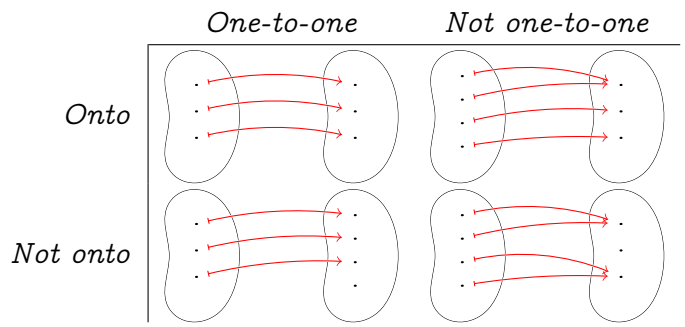
A *function* $f\colon V \to W$ is an association of inputs with outputs. We write $f(\vec{v}) = \vec{w}$ or $\vec{v} \mapsto \vec{w}$. The key property is that for each input $\vec{v}$ there is exactly one associate output $\vec{w}$ — that is, for each $\vec{v}$ there is an output $\vec{w}$ and there is only one such output. We say that the function is *well-defined*. To draw a function we sometimes use a bean diagram, with V on the left and W on the right, and with arrows connecting inputs with outputs.



A function is *one-to-one* if for every output there is at most one associated input. A function is *onto* if for every output there is at least one associated input.



To show that a function is one-to-one, the simplest thing is to assume that $f(\vec{v}_1) = f(\vec{v}_2)$ and go on to show that therefore $\vec{v}_1 = \vec{v}_2$. To show that a function is not one-to-one, the simplest thing is to find two different inputs $\vec{v}_1$ and $\vec{v}_2$ that map to the same output $f(\vec{v}_1) = f(\vec{v}_2)$.

To show that a function is onto, the simplest thing is to start with a $\vec{w}$ and give a formula for an input $\vec{v}$ that maps to it, that is, such that $f(\vec{v}) = \vec{w}$. To show that a function is not onto, the simplest thing is to produce a $\vec{w}$ for which there is no input $\vec{v}$ that maps to it, that is, such that there is no $\vec{v}$ with $f(\vec{v}) = \vec{w}$.

1. For the function $f\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ giving this association

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

what output column vector is associated with each input matrix?

a) $\begin{pmatrix} 5 & 2 \\ 1 & 9 \end{pmatrix}$

Solution: $\begin{pmatrix} 5 \\ 2 \\ 1 \ 9 \end{pmatrix}$

b) $\begin{pmatrix} 0 & -1/2 \\ 0 & 3 \end{pmatrix}$

Solution: $\begin{pmatrix} 0 \\ -1/2 \\ 0 \\ 3 \end{pmatrix}$

c) $\begin{pmatrix} x+y & 2z \\ x-y & y+z \end{pmatrix}$

Solution: $\begin{pmatrix} x+y \\ 2z \\ x-y \\ y+z \end{pmatrix}$

2. For the function $g\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ giving this association

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \\ b-d \\ d-b \end{pmatrix}$$

what output column vector is associated with each input matrix?

a) $\begin{pmatrix} 5 & 2 \\ 1 & 9 \end{pmatrix}$

Solution: $\begin{pmatrix} 5 \\ 1 \\ -7 \\ 7 \end{pmatrix}$

b) $\begin{pmatrix} 0 & -1/2 \\ 0 & 3 \end{pmatrix}$

Solution: $\begin{pmatrix} 0 \\ 0 \\ -7/2 \\ 7/2 \end{pmatrix}$

c) $\begin{pmatrix} x+y & 2z \\ x-y & y+z \end{pmatrix}$

Solution: $\begin{pmatrix} x+y \\ x-y \\ z-y \\ y-z \end{pmatrix}$

d) $\begin{pmatrix} a+2b & b+d \\ c+d & a+2b+d \end{pmatrix}$

Solution: $\begin{pmatrix} a+2b \\ c+d \\ a-b \\ b-a \end{pmatrix}$

3. Let $f\colon \mathcal{P}_2 \to \mathbb{R}^2$ be this.

$$f(c_2x^2 + c_1x + c_0) = \begin{pmatrix} -c_1 \\ 3c_0 \end{pmatrix}$$

a) Find $f(2x^2 - 4x + 3)$.

Solution: $\begin{pmatrix} 4 \\ 9 \end{pmatrix}$

b) Find $f(7)$.

Solution: $\begin{pmatrix} 0 \\ 21 \end{pmatrix}$

4. Consider $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ as here.

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a \\ a-b \end{pmatrix}$$

a) Find $f(\vec{v})$ for this vector.

$$\vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Solution: $f(\begin{pmatrix} 3 \\ 1 \end{pmatrix}) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

b) Show that f is one-to-one.

Solution: Suppose that $f(\vec{v}_1) = f(\vec{v}_2)$. Then

$$f(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}) = f(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix})$$

and so we have this.

$$\begin{pmatrix} a_1 \\ a_1 - b_1 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_2 - b_2 \end{pmatrix}$$

The top entries give that $a_1 = a_2$. The bottom entries give that $a_1 1 - b_1 = a_2 - b_2$, and with the information that $a_1 = a_2$, we have that $b_1 = b_2$ also. Therefore $\vec{v}_1 = \vec{v}_2$.

5. Consider $g\colon \mathcal{P}_2 \to \mathcal{P}_3$ given by $g(ax^2 + bx + c) = (a + b)x^3 - cx + b$.

a) Compute $g(9x^2 + 3x - 4)$.

Solution: $g(9x^2 + 3x - 4) = 12x^3 + 4x + 3$

b) Show that g is one-to-one.

Solution: Suppose that $g(\vec{v}_1) = g(\vec{v}_2)$. Writing $\vec{v}_1 = a_1x^2 + b_1x + c_1$ and $\vec{v}_2 = a_2x^2 + b_2x + c_2$ we get this.

$$(a_1 + b_1)x^3 - c_1x + b_1 = (a_2 + b_2)x^3 - c_2x + b_2$$

Comparing the two constant terms gives that $b_1 = b_2$. The linear terms give that $c_1 = c_2$. The coefficients of x^3 give $a_1 + b_1 = a_2 + b_2$, and since $b_1 = b_2$ we have $a_1 = a_2$ also. Thus $\vec{v}_1 = \vec{v}_2$.

6. Define $f\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^2$ as here.

$$f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a \\ a + b \end{pmatrix}$$

a) Find $f(\vec{v})$ for this input.

$$\vec{v} = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$$

Solution: $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

b) Show that f is not one-to-one.

Solution: Note that $f(\vec{v}_1) = f(\vec{v}_2)$ for these two.

$$\vec{v}_1 = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix}$$

7. Consider $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ as here.

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a \\ a-b \end{pmatrix}$$

a) Find $f(\vec{v})$ for this vector.

$$\vec{v} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

b) What vector $\vec{v}$ has $f(\vec{v}) = \vec{w}$?

$$\vec{w} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Solution:

$$\vec{v} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

c) What vector $\vec{v}$ has $f(\vec{v}) = \vec{w}$?

$$\vec{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Solution:

$$\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

d) Show that f is onto. That is, given an output $\vec{w}$, find a $\vec{v}$ such that $f(\vec{v}) = \vec{w}$.

Solution: Fix

$$\vec{w} = \begin{pmatrix} c \\ d \end{pmatrix}$$

and note that if

$$\vec{v} = \begin{pmatrix} c \\ c+d \end{pmatrix}$$

then $f(\vec{v}) = \vec{w}$.

8. Show that this function is onto.

$$f(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} a+2b \\ 3a+4b \end{pmatrix}$$

Solution: Fix an output.

$$\vec{w} = \begin{pmatrix} c \\ d \end{pmatrix}$$

We need to find an input

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$$

so that this holds.

$$\begin{pmatrix} a+2b \\ 3a+4b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

This is a linear system.

$$\begin{aligned} a+2b &= c \\ 3a+4b &= d \end{aligned} \xrightarrow{-3\rho_1+\rho_2} \begin{aligned} a+\ \ 2b &= c \\ -2b &= -3c+d \end{aligned}$$

We get $b = (3/2)c - (1/2)d$, and back substitution $a + 2 \cdot ((3/2)c - (1/2)d) = c$ leads to $a = -2c + d$. In short, where

$$\vec{v} = \begin{pmatrix} -2c+d \\ (3/2)c - (1/2)d \end{pmatrix}$$

then $f(\vec{v}) = \vec{w}$.

9. Show that this function is not onto.

$$g(ax^2 + bx + c) = \begin{pmatrix} a-b \\ 2a-2b \end{pmatrix}$$

Solution: The output vectors all have the property that the second component is twice the first. So this is not an output.

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$