Fix the vector spaces $V = \mathcal{P}_2$ and $W = \mathbb{R}^2$. Consider the map $h\colon V \to W$, which is a homomorphism.

$$h(ax^2 + bx + c) = \begin{pmatrix} a+b \\ a+c \end{pmatrix}$$

Consider also the member of the domain $\vec{v} = 3x^2 + 2x + 1$. Note that this is its image.

$$\vec{w} = h(\vec{v}) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

1. Fix these bases

$$B_1 = \langle x^2 + x, x + 1, x \rangle \qquad D_1 = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle$$

a) Represent $\vec{v}$ with respect to the first basis, $\mathrm{Rep}_{B_1}(\vec{v})$.

Solution: Solving this

$$3x^2 + 2x + 1 = c_1 \cdot (x^2 + x) + c_2 \cdot (x + 1) + c_3 \cdot (x)$$

gives this.

$$\mathrm{Rep}_{B_1}(\vec{v}) = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

b) Determine where each member of B_1 is mapped by h.

Solution: The basis vectors are mapped as here.

$$x^2 + x \mapsto \begin{pmatrix} 1+1 \\ 1+0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$x + 1 \mapsto \begin{pmatrix} 0+1 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$x \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

c) Represent each of those two-tall results with respect to D_1.

Solution: First is the calculation for $\mathrm{Rep}_{D_1}(h(\vec{\beta}_1))$.

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We get this.

$$\mathrm{Rep}_{D_1}(\begin{pmatrix}2\\1\end{pmatrix}) = \begin{pmatrix}3/2\\1/2\end{pmatrix}$$

The second one is the calculation for $\mathrm{Rep}_{D_1}(h(\vec{\beta}_2))$.

$$\begin{pmatrix}1\\1\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix}$$

We get this.

$$\mathrm{Rep}_{D_1}(\begin{pmatrix}1\\1\end{pmatrix}) = \begin{pmatrix}1\\0\end{pmatrix}$$

Third is the calculation for $\mathrm{Rep}_{D_1}(h(\vec{\beta}_3))$.

$$\begin{pmatrix}1\\0\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix}$$

We get this.

$$\mathrm{Rep}_{D_1}(\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}1/2\\1/2\end{pmatrix}$$

d) Give the matrix representing h with respect to the bases, $\mathrm{Rep}_{B_1,D_1}(h)$

Solution: Take the three column vectors from the prior item and make them the columns of a matrix, in order.

$$\mathrm{Rep}_{B_1,D_1}(h) = \begin{pmatrix}3/2 & 1 & 1/2\\1/2 & 0 & 1/2\end{pmatrix}$$

e) Use the matrix vector product definition to compute $\mathrm{Rep}_{B_1,D_1}(h)\cdot\mathrm{Rep}_{B_1}(\vec{v})$.

Solution: This matrix-vector multiplictation

$$\begin{pmatrix}3/2 & 1 & 1/2\\1/2 & 0 & 1/2\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-2\end{pmatrix}$$

gives this.

$$\begin{pmatrix}(3/2)\cdot 3 + 1\cdot 1 + (1/2)\cdot(-2)\\(1/2)\cdot 3 + 0\cdot 1 + (1/2)\cdot(-2)\end{pmatrix} = \begin{pmatrix}9/2\\1/2\end{pmatrix}$$

f) Verify that the result is $\mathrm{Rep}_{D_1}(h(\vec{v}))$.

Solution: Solving

$$\begin{pmatrix}5\\4\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix}$$

does indeed give $c_1 = 9/2$ and $c_2 = 1/2$.

2. Now fix these bases

$$B_2 = \langle x^2+x+1, x+1, 1\rangle \qquad D_2 = \langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\rangle$$

a) Find $\mathrm{Rep}_{B_2}(\vec{v})$ and $\mathrm{Rep}_{D_2}(\vec{w})$.

Solution: For $\mathrm{Rep}_{B_2}(\vec{v})$ we solve

$$c_1(x^2+x+1) + c_2(x+1) + c_3(1) = 3x^2+2x+1$$

to get this.

$$\mathrm{Rep}_{B_2}(\vec{v}) = \begin{pmatrix}3\\-1\\-1\end{pmatrix}$$

And for the other we solve

$$c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}5\\4\end{pmatrix}$$

leading to this.

$$\mathrm{Rep}_{D_2}(\vec{w}) = \begin{pmatrix}5\\-1\end{pmatrix}$$

b) Determine where each member of B_2 is mapped by h, and represent each of those two-tall results with respect to D_2.

Solution: This is the action of the map on the basis vectors from B_2.

$$x^2+x+1 \mapsto \begin{pmatrix}2\\2\end{pmatrix}$$

$$x+1 \mapsto \begin{pmatrix}1\\1\end{pmatrix}$$

$$x^2+x+1 \mapsto \begin{pmatrix}0\\1\end{pmatrix}$$

Representing the first with respect to D_2 means solving

$$\begin{pmatrix}2\\2\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix}$$

which gives this.

$$\mathrm{Rep}_{D_2}(h(\vec{\beta}_1)) = \begin{pmatrix}2\\0\end{pmatrix}$$

Representing $h(\vec{\beta}_2)$ with respect to D_2 means solving

$$\begin{pmatrix}1\\1\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix}$$

which gives this.

$$\mathrm{Rep}_{D_2}(h(x+1)) = \begin{pmatrix}1\\0\end{pmatrix}$$

And, representing $h(\vec{\beta}_3)$ with respect to D_2 means solving

$$\begin{pmatrix}0\\1\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix}$$

yielding this.

$$\mathrm{Rep}_{D_2}(h(1)) = \begin{pmatrix}0\\1\end{pmatrix}$$

c) Give $\mathrm{Rep}_{B_2,D_2}(h)$.

Solution: Take the three column vectors from the prior answer, in order, and make them the columns of the matrix.

$$\mathrm{Rep}_{B_2,D_2}(h) = \begin{pmatrix}2&1&0\\0&0&1\end{pmatrix}$$

d) Compute the matrix-vector product $\mathrm{Rep}_{B_2,D_2}(h)\cdot\mathrm{Rep}_{B_2}(\vec{v})$. Compare with $\mathrm{Rep}_{D_2}(\vec{w})$.

Solution: We have this.

$$\mathrm{Rep}_{B_2,D_2}(h)\cdot\mathrm{Rep}_{B_2}(\vec{v}) = \begin{pmatrix}2&1&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\-1\end{pmatrix} = \begin{pmatrix}5\\-1\end{pmatrix}$$