1. This is an upper-triangular matrix.

$$T = \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix}$$

We will diagonalize it.

a) Consider that the matrix represents a transformation $t\colon \mathbb{C}^2 \to \mathbb{C}^2$ with respect to the standard basis, $\mathcal{E}_2, \mathcal{E}_2$.

$$\mathcal{E} = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$$

Produce the appropriate commutative diagram, where the other basis is B.

Solution:

$$\begin{tikzcd}
\mathbb{C}^2_{wrt\ \mathcal{E}_2} \arrow[r, "t", "T"'] \arrow[d, "id", "P"'] & \mathbb{C}^2_{wrt\ \mathcal{E}_2} \arrow[d, "id", "P"'] \\
\mathbb{C}^2_{wrt\ B} \arrow[r, "t", "\hat{T}"'] & \mathbb{C}^2_{wrt\ B}
\end{tikzcd}$$

b) Set up the λ rescaling equations involving the vectors for the basis B.

Solution: Let $B = \langle \vec{\beta}_1, \vec{\beta}_2 \rangle$. By Lemma 2.4 we have that on the basis B the transformation rescales the basis vectors, that is, for some constants $\lambda_1, \lambda_2 \in \mathbb{C}$ we have this.

$$\begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix} \vec{\beta}_1 = \lambda_1 \vec{\beta}_1 \qquad \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix} \vec{\beta}_2 = \lambda_2 \vec{\beta}_2$$

c) Produce the resulting linear system.

Solution: We are looking for scalars $x \in \mathbb{C}$ so that this holds.

$$\begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = x \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

for some pairs $b_1, b_2 \in \mathbb{C}$. So: we want scalars $x \in \mathbb{C}$ so that this system has solution

$$\begin{aligned} 2b_1 + \quad b_2 &= x \cdot b_1 \\ -3b_2 &= x \cdot b_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} (2-x)b_1 + \qquad b_2 &= 0 \\ (-3-x)b_2 &= 0 \end{aligned}$$

for some $b_1, b_2 \in \mathbb{C}$ that are not both zero (since the vector with both entries zero can not be a member of any basis).

d) Solve that linear system.

Solution: Start with

$$\begin{aligned}(2-x)b_1 + \qquad b_2 &= 0\\ (-3-x)b_2 &= 0\end{aligned}$$

and focus first on the bottom equation. Either $x = -3$ or $b_2 = 0$.

Following up on the second case, if $b_2 = 0$ then the first equation becomes $(2-x)b_1 = 0$. We can't have $b_1 = 0$ also, since we can't have that both b's are zeros. So we get the other solution $x = 2$.

We write these as $\lambda_1 = 2$ and $\lambda_2 = -3$.

e) Produce the vector in the resulting basis B associated with $\lambda_1 = 2$.

Solution: To get $\vec{\beta}_1$, the vector associated with $\lambda_1 = 2$, we have this.

$$\begin{pmatrix}2 & 1\\ 0 & -3\end{pmatrix}\vec{\beta}_1 = \lambda_1\vec{\beta}_1 \quad \Rightarrow \begin{pmatrix}2 & 1\\ 0 & -3\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix} = 2\cdot b_1\vec{b}_2$$

It gives a linear system.

$$\begin{aligned}2b_1 + \quad b_2 &= 2b_1\\ -3b_2 &= 2b_2\end{aligned}$$

We conclude that $b_2 = 0$ and b_1 is free. That is, the set of associated vector solutions is this.

$$V_2 = \{\begin{pmatrix}b_1\\ 0\end{pmatrix} \mid b_1 \in \mathbb{C}\}$$

For the frist basis vector $\vec{\beta}_1$ we can pick any nonzero member of that set. This will do.

$$\vec{\beta}_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$$

f) Also produce the vector in B associated with $\lambda_2 = -3$.

Solution: To get $\vec{\beta}_2$, the vector associated with $\lambda_2 = -3$, we have this.

$$\begin{pmatrix}2 & 1\\ 0 & -3\end{pmatrix}\vec{\beta}_2 = \lambda_2\vec{\beta}_2 \quad \Rightarrow \begin{pmatrix}2 & 1\\ 0 & -3\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix} = -3\cdot b_1\vec{b}_2$$

It gives a linear system.

$$\begin{aligned}2b_1 + \quad b_2 &= -3b_1\\ -3b_2 &= -3b_2\end{aligned}$$

We conclude that b_2 is free. We take it as a parameter, rewrite the first equation as

$5b_1 + b_2 = 0$, and the set of associated vector solutions is this.

$$V_{-3} = \{ \begin{pmatrix} (-1/5)b_2 \\ b_2 \end{pmatrix} \mid b_2 \in \mathbb{C} \}$$

For $\vec{\beta}_2$ we can pick any nonzero member. This will do.

$$\vec{\beta}_2 = \begin{pmatrix} -1/5 \\ 1 \end{pmatrix}$$

g) Exhibit the basis B.

Solution:

$$B = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/5 \\ 1 \end{pmatrix} \rangle$$

h) Check the change of basis.

Solution: Here is the diagram again.

$$\begin{array}{ccc} \mathbb{C}^2_{wrt\ \mathcal{E}_2} & \xrightarrow[T]{t} & \mathbb{C}^2_{wrt\ \mathcal{E}_2} \\ {\scriptstyle id}\downarrow{\scriptstyle P} & & {\scriptstyle id}\downarrow{\scriptstyle P} \\ \mathbb{C}^2_{wrt\ B} & \xrightarrow[\hat{T}]{t} & \mathbb{C}^2_{wrt\ B} \end{array}$$

We know the matrices on the top and bottom.

$$T = \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix} \qquad \hat{T} = \begin{pmatrix} 2 & \\ 0 & -3 \end{pmatrix}$$

The matrix on the right is $P = \mathrm{Rep}_{id}(B, \mathcal{E}_2)$. To find it we compute the action of the function on the starting basis

$$\vec{\beta}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xmapsto{id} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{\beta}_2 = \begin{pmatrix} -1/5 \\ 1 \end{pmatrix} \xmapsto{id} \begin{pmatrix} -1/5 \\ 1 \end{pmatrix}$$

and represent each result with respect to the ending basis,

$$\mathrm{Rep}_{\mathcal{E}_2}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_2}(\begin{pmatrix} -1/5 \\ 1 \end{pmatrix}) = \begin{pmatrix} -1/5 \\ 1 \end{pmatrix}$$

to give this.

$$P = \begin{pmatrix} 1 & -1/5 \\ 0 & 1 \end{pmatrix}$$

For the matrix on the left side P^{-1}, we use the formula for the inverse of a 2×2 matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad P^{-1} = \frac{1}{1} \cdot \begin{pmatrix} 1 & 1/5 \\ 0 & 1 \end{pmatrix}$$

Finally, we get a diagonal matrix.

$$\hat{T} = P^{-1}TP = \begin{pmatrix} 1 & 1/5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & -1/5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$