1. The first chapter informally describes that the bottom row in this matrix is "redundant" in a linear sense because twice the first row plus the second equals the third.

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 2 & 3 & 5 & 2 \end{pmatrix}$$

We now have the formal terminology: this set is linearly dependent.

$$\{(1\ \ 0\ \ 2\ \ 1), (0\ \ 3\ \ 1\ \ 0), (2\ \ 3\ \ 5\ \ 2)\}$$

Verify that by determining whether there is a linear relationship among the row vectors, whether there are coefficients for $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ other than the trivial ones, $c_1 = c_2 = c_3 = 0$.

Solution: From

$$c_1 \cdot (1\ \ 0\ \ 2\ \ 1) + c_2 \cdot (0\ \ 3\ \ 1\ \ 0) + c_3 \cdot (2\ \ 3\ \ 5\ \ 2) = (0\ \ 0\ \ 0\ \ 0)$$

the relationship among first entries, second entries, etc., gives this.

$$\begin{aligned} c_1 \qquad\quad + 2c_3 &= 0 \\ 3c_2 + 3c_3 &= 0 \\ 2c_1 + \ c_2 + 5c_3 &= 0 \\ c_1 \qquad\quad + 2c_3 &= 0 \end{aligned}$$

Gauss's method does not give a unique solution, so the set is linearly dependent.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow[-1\rho_1+\rho_4]{-2\rho_1+\rho_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{(-1/3)\rho_2+\rho_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2. Find whether this subset of $\mathcal{P}_2$ is linearly dependent or linearly independent.

$$\{x^2 - x + 1,\ 3x + 4,\ 2x^2 - x\}$$

Solution: Looking for a linear relationship

$$c_1(x^2 - x + 1) + c_2(3x + 4) + c_3(2x^2 - x) = 0x^2 + 0x + 0$$

leads to this linear system.

$$\begin{aligned} c_1 \qquad\quad + 2c_3 &= 0 \\ -c_1 + 3c_2 - \ c_3 &= 0 \\ c_1 + 4c_2 \qquad\quad &= 0 \end{aligned}$$

Gauss's method tells us that the only solution is the trivial one.

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & 3 & -1 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right) \xrightarrow[-1\rho_1+\rho_3]{1\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 4 & -2 & 0 \end{array}\right) \xrightarrow{(-4/3)\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -10/3 & 0 \end{array}\right)$$

So the set is linearly independent.

3. Is this set linearly independent?

$$\{\begin{pmatrix} 0 & 4 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\}$$

Solution: Set up

$$c_1 \cdot \begin{pmatrix} 0 & 4 \\ -2 & 0 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

to get this linear system.

$$\begin{aligned} 3c_2 + c_3 &= 0 \\ 4c_1 + \ c_2 \qquad &= 0 \\ -2c_1 + \ c_2 + c_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Apply Gauss's method.

$$\left(\begin{array}{ccc|c} 0 & 3 & 1 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_2} \left(\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{(1/2)\rho_1+\rho_3} \left(\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{(-1/2)\rho_2+\rho_3} \left(\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

There is a unique solution, so the set is linearly independent.