The determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and this is the determinant of a 3×3 matrix.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - hfa - idb - gec$$

1a) Perform Gauss's method on the matrix, decide if it is nonsingular, and then compute the determinant.

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Solution: Here is Gauss's method.

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \xrightarrow{-2\rho_1+\rho_2} \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix}$$

Because there are no zero rows, the starting matrix is nonsingular. Here is the determinant.

$$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 1\cdot 4 - 3\cdot 2 = 4 - 6 = -2$$

b) Do the same to this matrix.

$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

Solution: Here is Gauss's method.

$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \xrightarrow{-2\rho_1+\rho_2} \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$$

Because the is a zero row, the starting matrix is singular. Here is the determinant.

$$\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 1\cdot 6 - 3\cdot 2 = 0$$

c) Perform Gauss's method on the matrix, decide if it is nonsingular, and then compute the determinant.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Solution: Gauss's method is straightforward.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow[-7\rho_1+\rho_3]{-4\rho_1+\rho_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \xrightarrow{-2\rho_2+\rho_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

There are zero rows so the starting matrix is singular. Here is the determinant.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\cdot 5\cdot 9 + 2\cdot 6\cdot 7 + 3\cdot 4\cdot 8 - 7\cdot 5\cdot 3 - 8\cdot 6\cdot 1 - 9\cdot 4\cdot 2 = 0$$

d) Do the same to this matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$$

Solution: Gauss's method is much the same as in the prior answer.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} \xrightarrow[-7\rho_1+\rho_3]{-4\rho_1+\rho_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{pmatrix} \xrightarrow{-2\rho_2+\rho_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{pmatrix}$$

There are zero rows so the starting matrix is singular. Here is the determinant.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = 1\cdot 5\cdot 10 + 2\cdot 6\cdot 7 + 3\cdot 4\cdot 8 - 7\cdot 5\cdot 3 - 8\cdot 6\cdot 1 - 10\cdot 4\cdot 2 = -3$$

2. Find the determinant of each matrix by reducing it to echelon form, keeping track of any row swaps, and then multiplying down the diagonal.

a)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Solution: Gauss's method is straightforward.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow[-7\rho_1+\rho_3]{-4\rho_1+\rho_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \xrightarrow{-2\rho_2+\rho_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

Multiplying down the diagonal gives 0.

b)

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & -1 & 0 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 0 & 0 \end{pmatrix}$$

Solution: Gauss's method is straightforward.

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & -1 & 0 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 0 & 0 \end{pmatrix} \xrightarrow[\rho_1+\rho_4]{-2\rho_1+\rho_3} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 2 & 2 & -1 \end{pmatrix}$$

$$\xrightarrow[-\rho_2+\rho_4]{-(1/2)\rho_2+\rho_3} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -7/2 & 4 \\ 0 & 0 & 3 & -1 \end{pmatrix}$$

$$\xrightarrow{(6/7)\rho_3+\rho_4} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -7/2 & 4 \\ 0 & 0 & 0 & 17/7 \end{pmatrix}$$

Multiplying down the diagonal gives -17.