1. For each space, find a basis. Verify that it is a basis.

a) $S_1 = \{ax^2 + bx + c \mid a - c = 0\}$

b) $S_2 = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + 2y = 0 \text{ and } x + y - z = 0\}$

Solution:

a) We have $S_1 = \{cx^2 + bx + c \mid b, c \in \mathbb{R}\}$. That equals $\{c \cdot (x^1 + 1) + b \cdot (x) \mid b, c \in \mathbb{R}\}$. A spanning set is $\{x^2 + 1, x\}$.

It is linearly independent because the relationship

$$c_0(x^2 + 1) + c_1(x) = 0x^2 + 0x + 0$$

gives that $c_0 = c_1 = 0$. Thus it is a basis for S_1.

b) The two restrictions give this linear system.

$$\begin{array}{l} x + 2y \quad\;\; = 0 \\ x + \;\; y - z = 0 \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{l} x + \;\; 2y \quad\;\; = 0 \\ \qquad\quad -y - z = 0 \end{array}$$

Leading are x and y. Free is z. Expressing the leading variables in terms of the free variable gives this.

$$S_2 = \{\begin{pmatrix} 2z \\ -z \\ z \end{pmatrix} \mid z \in \mathbb{R}\} = \{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot z \mid z \in \mathbb{R}\}$$

So this is a spanning set.

$$\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\}$$

To check that this set is a basis, we need only check that it is linearly independent. A set with only one vector is linearly independent unless that single vector is the zero vector, which this one isn't. So it is a basis.

2. For the vector space

$$V = \{ax^2 + bx + c \mid a - c = 0\}$$

this set is a basis.

$$B = \{x^2 + 1, x\}$$

a) Where $\vec{v} = 3x^2 + 2x + 3$, find $\mathrm{Rep}_B(\vec{v})$.

b) Find $\mathrm{Rep}_B(-x^2 + 4x - 1)$.

Solution:

a) We find what coefficients there are in this relationship.

$$3x^2 + 2x + 3 = c_1 \cdot (x + 2 + 1) + c_2(x)$$

Obviously, $c_1 = 3$ and $c_2 = 2$ work here.

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

b) We find what coefficients there are in this relationship.

$$-x^2 + 4x - 1 = c_1 \cdot (x + 2 + 1) + c_2(x)$$

By eye, $c_1 = -1$ and $c_2 = 4$.

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

3. This is a basis for the xy plane.

$$B = \{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \}$$

a) Draw the plane, and plot the basis vectors. Also draw the line through each.

b) Draw

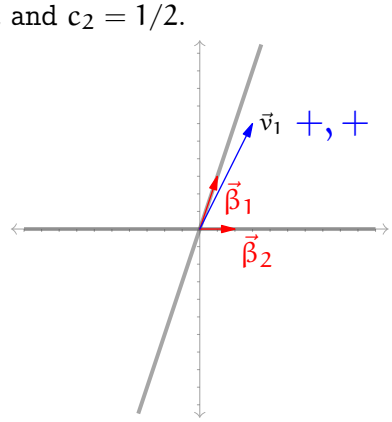
$$\vec{v}_1 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

and find $\mathrm{Rep}_B(\vec{v}_1)$. Label $\vec{v}_1$ with the signs of the coefficients, such as "$+,+$" or "$+,-$".

Solution: From

$$\begin{pmatrix} 3 \\ 6 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

we can see by eye that $c_1 = 2$ and $c_2 = 1/2$.

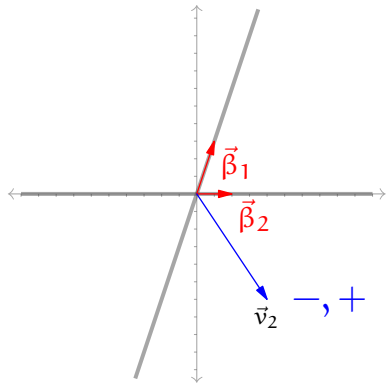


c) Do the same for

$$\vec{v}_2 = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

Solution: Again by eye, we have

$$\mathrm{Rep}_B(\vec{v}_2) = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

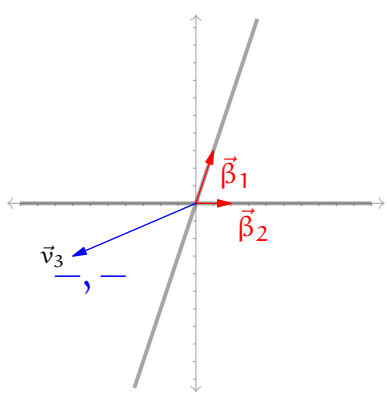


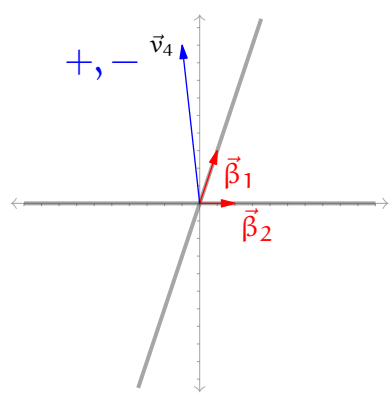
d) Repeat for

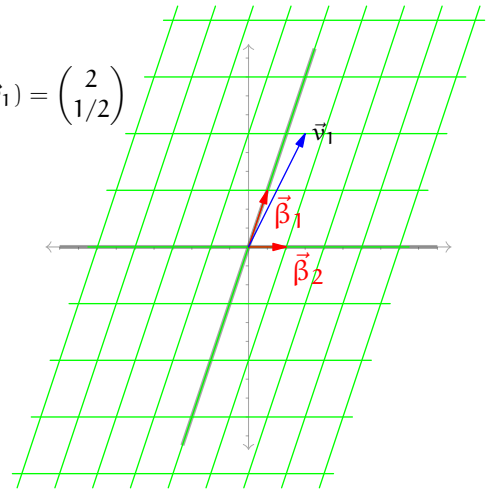
$$\vec{v}_3 = \begin{pmatrix} -7 \\ -3 \end{pmatrix} \qquad \vec{v}_4 = \begin{pmatrix} -1 \\ 9 \end{pmatrix}$$

Solution: We have

$$\mathrm{Rep}_B(\vec{v}_3) = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \qquad \mathrm{Rep}_B(\vec{v}_4) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$





Overall, this basis for the plane defines a coordinate system.

$$\mathrm{Rep}_B(\vec{v}_1) = \begin{pmatrix} 2 \\ 1/2 \end{pmatrix}$$