1. Recall from last time when we diagonalized this upper-triangular matrix

$$T = \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix}$$

that our use of the lemma led to this linear system.

$$\begin{aligned} (2-x)b_1 + \qquad\qquad b_2 &= 0 \\ (-3-x)b_2 &= 0 \end{aligned}$$

a) Take $x = -3$ and solve the linear system. The set of solutions forms a subspace. It is the *eigenspace* V_{-3} *associated with the eigenvalue* $\lambda = -3$.

Solution:

$$\begin{aligned} 2b_1 + \quad b_2 &= -3b_1 \\ -3b_2 &= -3b_2 \end{aligned}$$

We conclude that b_2 is free. We take it as a parameter, rewrite the first equation as $5b_1 + b_2 = 0$, and the set of associated vector solutions is this.

$$V_{-3} = \{ \begin{pmatrix} (-1/5)b_2 \\ b_2 \end{pmatrix} \mid b_2 \in \mathbb{C} \}$$

b) Take $x = 2$ and solve the linear system to find V_2.

Solution: It gives a linear system.

$$\begin{aligned} 2b_1 + \quad b_2 &= 2b_1 \\ -3b_2 &= 2b_2 \end{aligned}$$

We conclude that $b_2 = 0$ and b_1 is free. That is, the set of associated vector solutions is this.

$$V_2 = \{ \begin{pmatrix} b_1 \\ 0 \end{pmatrix} \mid b_1 \in \mathbb{C} \}$$

c) $x = 4$

Solution: Plugging $x = 4$ into this system

$$\begin{aligned} (2-x)b_1 + \qquad\qquad b_2 &= 0 \\ (-3-x)b_2 &= 0 \end{aligned}$$

gives this matrix equation.

$$\begin{pmatrix} -2 & 1 \\ 0 & -7 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The matrix is nonsingular, so there is a unique solution. Because the system is homogeneous, that solution is this.

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

d) $x = 0$

Solution: Plugging $x = 0$ into

$$\begin{aligned} (2-x)b_1 + \qquad\quad b_2 &= 0 \\ (-3-x)b_2 &= 0 \end{aligned}$$

gives this matrix equation.

$$\begin{pmatrix} 2 & 1 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The matrix is nonsingular, so there is a unique solution.

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2. We will use the determinant function to solve $T\vec{v} = x\vec{v}$.

$$\begin{pmatrix} 13 & -4 \\ -4 & 7 \end{pmatrix}$$

a) Set up the matrix equation and bring all variables to the left side.

Solution: This is the matrix equation.

$$T\vec{v} = x \cdot \vec{v}$$

$$\begin{pmatrix} 13 & -4 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = x \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

We get this linear system.

$$\begin{aligned} 13b_1 - 4b_2 &= x \cdot b_1 \\ -4b_1 + 7b_2 &= x \cdot b_2 \end{aligned}$$

Bring all the variables to the left side.

$$\begin{aligned} (13-x)b_1 - \qquad\quad 4b_2 &= 0 \\ -4b_1 + (7-x)b_2 &= 0 \end{aligned}$$

This is the result.

$$\begin{pmatrix} 13-x & -4 \\ -4 & 7-x \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$(T - x \cdot I)\vec{v} = \vec{0}$$

b) Find the *characteristic polynomial*, $|T - xI|$.

Solution: Apply the $ad - bc$ determinant formular to get this.

$$\begin{pmatrix} 13-x & -4 \\ -4 & 7-x \end{pmatrix} = (13-x)(7-x) - (-4)(-4) = x^2 - 20x + 75 = (x-5)(x-15)$$

c) The factors of the characteristic equation $|T - xI| = 0$ are 5 and 15. Find the eigenspace associated with $\lambda_1 = 5$.

Solution: Into this equation, plug $x = 5$.

$$\begin{pmatrix} 13-x & -4 \\ -4 & 7-x \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We get a linear system.

$$\begin{aligned} 8b_1 - 4b_2 &= 0 \\ -4b_1 + 2b_2 &= 0 \end{aligned} \quad \xrightarrow{(1/2)\rho_1 + \rho_2} \quad \begin{aligned} 8b_1 - 4b_2 &= 0 \\ 0 \quad &= 0 \end{aligned}$$

Leading is b_1 and free is b_2, so $b_1 = (1/2) \cdot b_2$.

$$V_5 = \{ \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \cdot b_2 \mid b_2 \in \mathbb{C} \}$$

d) Find the eigenspace associated with $\lambda_2 = 15$.

Solution: Into this equation, plug $x = 15$.

$$\begin{pmatrix} 13-x & -4 \\ -4 & 7-x \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We get a linear system.

$$\begin{aligned} -2b_1 - \quad 4b_2 &= 0 \\ -4b_1 + -8b_2 &= 0 \end{aligned} \quad \xrightarrow{-2\rho_1 + \rho_2} \quad \begin{aligned} -2b_1 - 4b_2 &= 0 \\ 0 \quad &= 0 \end{aligned}$$

Leading is b_1 and free is b_2, so $b_1 = -2 \cdot b_2$.

$$V_{15} = \{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot b_2 \mid b_2 \in \mathbb{C} \}$$