In $\mathbb{R}^n$ the *box* (or *parallelepiped*) formed by $\langle \vec{v}_1, \ldots, \vec{v}_n \rangle$ is this set

$$\{t_1\vec{v}_1 + \cdots + t_n\vec{v}_n \mid t_1, \ldots, t_n \in [0 \ldots 1]\}$$

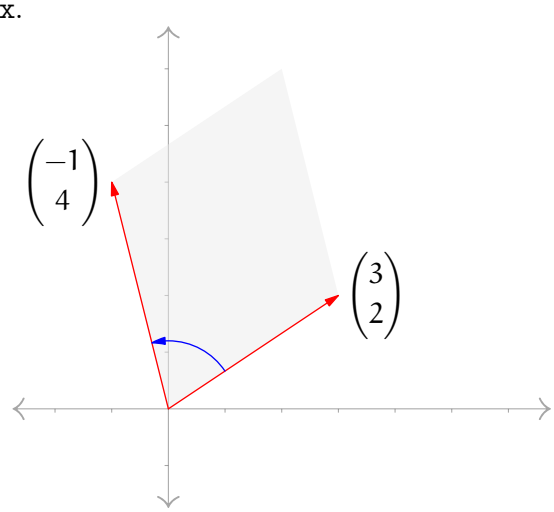
(this is like the span but the coefficient t_i's are restricted to being in the interval $[0 \ldots 1]$).

The *size* of a box is the determinant of the matrix with the vectors as columns. The absolute value of the size is the box's volume, and the sign is its orientation.

1. Draw the box defined by each list of vectors. Find its size, and its volume and orientation.

a) $\langle \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \end{pmatrix} \rangle$

Solution: Here is the box.



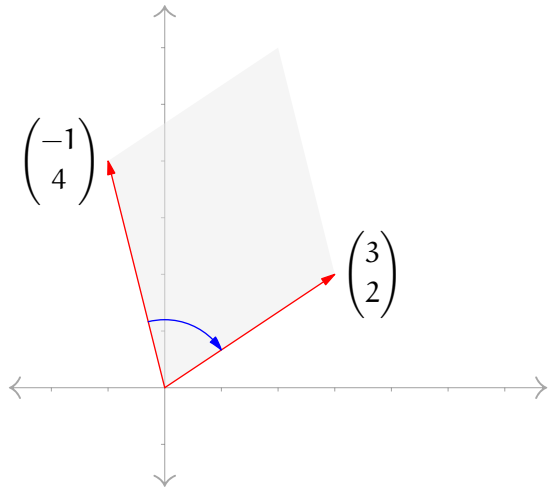
The size is the determinant.

$$\begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix} = 3 \cdot 4 - (-1) \cdot 2 = 14$$

The volume is the absolute value of the size, so volume $= 14$, and the orientation is positive, '+'. Note the meaning of that orientation: to get from the first vector to the second requires going counterclockwise, as with the blue arrow.

b) $\langle \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \rangle$

Solution: Here is the box, and the size.

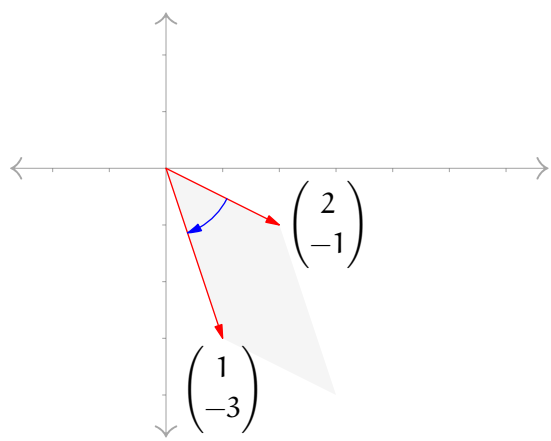


$$\begin{vmatrix}-1 & 3\\4 & 2\end{vmatrix} = -14$$

The volume is the absolute value of the size, so volume $= 14$, and the orientation is negative, '$-$'. Here, to get from the first vector to the second, with the blue arrow, requires going clockwise.

c) $\left\langle \begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}1\\-3\end{pmatrix} \right\rangle$

Solution: The box is different, and the size.

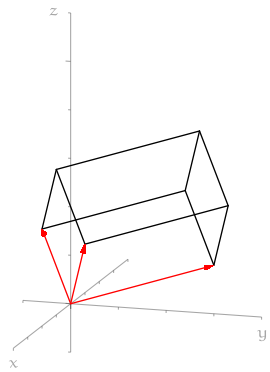


$$\begin{vmatrix}2 & 1\\-1 & -3\end{vmatrix} = -5$$

The volume is volume $= 5$, and the orientation is negative, '$-$'.

d) $\left\langle \begin{pmatrix}2\\0\\2\end{pmatrix}, \begin{pmatrix}0\\3\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix} \right\rangle$

Solution: Here is the box and the size.



$$\begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 2 & 1 & 1 \end{vmatrix} = 12$$

The volume is volume $= 12$ and the orientation is positive, '+'.

2. Consider the map $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ represented with respect to the standard basis by this matrix.

$$T = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$$

a) Compute the effect of that map, $t(\vec{v}_1) = \vec{w}_1$ and $t(\vec{v}_2) = \vec{w}_2$, on the two vectors in this sequence.

$$\langle \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} \rangle$$

Solution: We are doing the computation $\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(\vec{v}) \cdot \mathrm{Rep}_{\mathcal{E}_2}(\vec{v}) = \mathrm{Rep}_{\mathcal{E}_2}(t(\vec{v}))$.

$$\begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

The other is similar.

$$\begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

b) Do the same for this sequence. $\langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix} \rangle$

Solution:

$$\begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

c) For the results of the two prior parts, compute the size of the box defined by $\langle \vec{w}_1, \vec{w}_2 \rangle$.

Solution:

$$\begin{vmatrix} 5 & 3 \\ -6 & 2 \end{vmatrix} = 28 \qquad \begin{vmatrix} 1 & -2 \\ -4 & -2 \end{vmatrix} = -10$$

d) For those two cases, take the ratios size of the output box/size of input box.

Solution: We have this.

Input box	*Size*	*Output box*	*Size*
$\langle \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \end{pmatrix} \rangle$	14	$\langle \begin{pmatrix} 5 \\ -6 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \rangle$	28
$\langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix} \rangle$	-5	$\langle \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix} \rangle$	-10

Of course, they double. Note this.

$$\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = 2$$