1. Consider the plane, $\mathbb{R}^2$, and its canonical basis.

$$\mathcal{E}_2 = \langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \rangle$$

We can specify a linear map from $\mathbb{R}^2$ to itself, that is, a linear transformation of $\mathbb{R}^2$, by specifying its action on the basis vectors.

$$\begin{pmatrix}1\\0\end{pmatrix} \xmapsto{h} \begin{pmatrix}1\\1\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix} \xmapsto{h} \begin{pmatrix}-2\\3\end{pmatrix}$$

a) Find the value of the function on this input.

$$h(\begin{pmatrix}4\\5\end{pmatrix})$$

Solution: Expand the given vector in terms of the standard basis.

$$h(\begin{pmatrix}4\\5\end{pmatrix}) = h(\ 4\cdot\begin{pmatrix}1\\0\end{pmatrix} + 5\cdot\begin{pmatrix}0\\1\end{pmatrix}\ )$$

Apply that the map preserves addition

$$= h(4\cdot\begin{pmatrix}1\\0\end{pmatrix}) + h(5\cdot\begin{pmatrix}0\\1\end{pmatrix})$$

and scalar multiplication.

$$= 4\cdot h(\begin{pmatrix}1\\0\end{pmatrix}) + 5\cdot h(\begin{pmatrix}0\\1\end{pmatrix})$$

Finish by evaluating the map on the basis vectors.

$$= 4\cdot\begin{pmatrix}1\\1\end{pmatrix} + 5\cdot\begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}-6\\19\end{pmatrix}$$

b) Also find the output on this input.

$$h(\begin{pmatrix}-7\\0\end{pmatrix})$$

Solution: Go through the same steps as in the prior part.

$$\begin{aligned}
h(\begin{pmatrix}-7\\0\end{pmatrix}) &= h(\ -7\cdot\begin{pmatrix}1\\0\end{pmatrix}+0\cdot\begin{pmatrix}0\\1\end{pmatrix}\ )\\
&= h(-7\cdot\begin{pmatrix}1\\0\end{pmatrix})+h(0\cdot\begin{pmatrix}0\\1\end{pmatrix})\\
&= -7\cdot h(\begin{pmatrix}1\\0\end{pmatrix})+0\cdot h(\begin{pmatrix}0\\1\end{pmatrix})\\
&= -7\cdot\begin{pmatrix}1\\1\end{pmatrix}+0\cdot\begin{pmatrix}-2\\3\end{pmatrix}\\
&= \begin{pmatrix}-7\\-7\end{pmatrix}
\end{aligned}$$

c) Find the general formula for this map.

$$h(\begin{pmatrix}x\\y\end{pmatrix})$$

Solution: We have this.

$$\begin{aligned}
h(\begin{pmatrix}x\\y\end{pmatrix}) &= h(\ x\cdot\begin{pmatrix}1\\0\end{pmatrix}+y\cdot\begin{pmatrix}0\\1\end{pmatrix}\ )\\
&= h(x\cdot\begin{pmatrix}1\\0\end{pmatrix})+h(y\cdot\begin{pmatrix}0\\1\end{pmatrix})\\
&= x\cdot h(\begin{pmatrix}1\\0\end{pmatrix})+y\cdot h(\begin{pmatrix}0\\1\end{pmatrix})\\
&= x\cdot\begin{pmatrix}1\\1\end{pmatrix}+y\cdot\begin{pmatrix}-2\\3\end{pmatrix}\\
&= \begin{pmatrix}x-2y\\x+3y\end{pmatrix}
\end{aligned}$$

2. Consider the vector space of quadratic polynomials.

$$\mathcal{P}_2 = \{\, ax^2+bx+c \mid a,b,c\in\mathbb{R}\,\}$$

This is a basis: $B=\{x^2+1, x^2, 3x-1\}$. Define the the transformation $f\colon \mathcal{P}_2\to\mathcal{P}_2$ by starting

with this action on the elements of the basis B, and extending linearly.

$$x^2+1 \mapsto 2x \qquad x^2 \mapsto 2x \qquad 3x-1 \mapsto 3$$

a) What is $f(4x^2-2x+5)$?

Solution: First get an expression for $4x^2-2x+5$ as a linear combination of basis elements. This equation

$$4x^2-2x+5 = c_1(x^2+1)+c_2(x^2)+c_3(3x-1)$$

gives this linear system.

$$\begin{aligned} c_1+c_2 \qquad &= 4 \\ 3c_3 &= -2 \\ c_1 \qquad - c_3 &= 5 \end{aligned}$$

We can solve it by eye, giving $c_3=-2/3$, then $c_1=13/3$, and then $c_2=-1/3$. Restated, we have this.

$$\mathrm{Rep}_B(4x^2-2x+5) = \begin{pmatrix} 13/3 \\ -1/3 \\ -2/3 \end{pmatrix}$$

Now proceed as in the prior question.

$$\begin{aligned} f(4x^2-2x+5) &= f(\ \frac{13}{3}\cdot(x^2+1)+\frac{-1}{3}\cdot(x^2)+\frac{-2}{3}\cdot(3x-1)\ ) \\ &= f(\frac{13}{3}\cdot(x^2+1))+f(\frac{-1}{3}\cdot(x^2))+f(\frac{-2}{3}\cdot(3x-1)) \\ &= \frac{13}{3}\cdot f(x^2+1)+\frac{-1}{3}\cdot f(x^2)+\frac{-2}{3}\cdot f(3x-1) \\ &= \frac{13}{3}\cdot(2x)+\frac{-1}{3}\cdot(2x)+\frac{-2}{3}\cdot(3) \\ &= 8x-2 \end{aligned}$$

b) $f(-3x+2)$?

Solution: First,

$$-3x+2 = c_1(x^2+1)+c_2(x^2)+c_3(3x-1)$$

gives this.

$$\begin{aligned} c_1+c_2 \qquad &= 0 \\ 3c_3 &= -3 \\ c_1 \qquad - c_3 &= 2 \end{aligned}$$

We get $c_3=-1$, then $c_1=1$, and then $c_2=-1$.

$$\mathrm{Rep}_B(-3x+2) = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Now just apply linearity.

$$\begin{aligned}
f(-3x+2) &= f(\ 1\cdot(x^2+1) - 1\cdot(x^2) - 1\cdot(3x-1)\ ) \\
&= f(1\cdot(x^2+1)) + f(-1\cdot(x^2)) + f(-1\cdot(3x-1)) \\
&= f(x^2+1) - f(x^2) - f(3x-1) \\
&= (2x) - (2x) + -(3) \\
&= -3
\end{aligned}$$

c) Next we find the formula for $f(ax^2+bx+c)$. First, represent ax^2+bx+c with respect to the basis.

Solution: From

$$ax^2+bx+c = c_1(x^2+1) + c_2(x^2) + c_3(3x-1)$$

we get this.

$$\begin{aligned}
c_1 + c_2 \qquad\quad &= a \\
3c_3 &= b \\
c_1 \qquad - \ c_3 &= c
\end{aligned}$$

We get $c_3 = b/3$, then $c_1 = (b/3)+c$, and then $c_2 = a-(b/3)-c$.

$$\mathrm{Rep}_B(-3x+1) = \begin{pmatrix} (b/3)+c \\ a-(b/3)-c \\ b/3 \end{pmatrix}$$

d) Finish by applying linearity.

Solution: Using the prior part we have this.

$$\begin{aligned}
f(ax^2+bx+c) &= f(\ ((b/3)+c)\cdot(x^2+1) + (a-(b/3)-c)\cdot(x^2) + (b/3)\cdot(3x-1)\ ) \\
&= ((b/3)+c)\cdot f(x^2+1) + (a-(b/3)-c)\cdot f(x^2) + (b/3)\cdot f(3x-1) \\
&= ((b/3)+c)\cdot(2x) + (a-(b/3)-c)\cdot(2x) + (b/3)\cdot(3) \\
&= 2a\cdot x + b
\end{aligned}$$

e) What name do you know this map by?

Solution: This is the derivative.