1. Find the dimension of each space.

a) $V_1 = \{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a - b + c = 0 \text{ and } a - 2c = 0 \}$

Solution: We need a basis. For a spanning set we parametrize.

$$\begin{array}{rcl} a - b + \ \ c = 0 & \xrightarrow{-\rho_1+\rho_2} & a - b + \ \ c = 0 \\ a \qquad - 2c = 0 & & b - 3c = 0 \end{array}$$

We get $b = 3c$ and $a = 2c$.

$$\begin{aligned} V_1 &= \{ \begin{pmatrix} 2c & 0 \\ 3c & c \end{pmatrix} \mid c \in \mathbb{R} \} \\ &= \{ \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \cdot c \mid c \in \mathbb{R} \} \end{aligned}$$

The spanning set is this.

$$\{ \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \}$$

Because it only has one element, and that element is not the zero element, this set is linearly independent. So it is a basis. The dimension is the number of elements in that basis, one.

b) $V_2 = \{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a - b + 3c = 0 \}$

Solution: We have this.

$$V_2 = \{ \begin{pmatrix} b - 3c \\ b \\ c \end{pmatrix} \mid b, c \in \mathbb{R} \} = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot b + \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \cdot c \mid b, c \in \mathbb{R} \}$$

So this is a spanning set.

$$B = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \}$$

To verify that it is linearly independent, set up a relationship.

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Looking at the second and third components shows that the only solution is the trivial one. So the set B is a basis and the dimension is 2.

c) $V_3 = \{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a - b + 3c = 0 \text{ and } 2a - c = 0 \}$

Solution: Reduce the linear system.

$$\begin{array}{rcl} a - b + 3c = 0 \\ 2a \quad - \ c = 0 \end{array} \xrightarrow{-2\rho_1+\rho+2} \begin{array}{rcl} a - \ b + 3c = 0 \\ 2b - 7c = 0 \end{array}$$

That leads to $b = (7/2)c$ and $a = (1/2)c$. We have this.

$$V_3 = \{ \begin{pmatrix} (1/2)c \\ (7/2)c \\ c \end{pmatrix} \mid c \in \mathbb{R} \} = \{ \begin{pmatrix} 1/2 \\ 7/2 \\ 1 \end{pmatrix} \cdot c \mid c \in \mathbb{R} \}$$

This is the spanning set.

$$\{ \begin{pmatrix} 1/2 \\ 7/2 \\ 1 \end{pmatrix} \}$$

Because it only has one (non-zero) element, it is linearly independent, and so is a basis. The dimension is 1.

d) $V_4 = \{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a - b + 3c = 0 \text{ and } 2a - c = 0 \text{ and } a + b - 4c = 0 \}$

Solution: The linear reduction gives this.

$$\begin{array}{rcl} a - b + 3c = 0 \\ 2a \quad - \ c = 0 \\ a + b - 4c = 0 \end{array} \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{array}{rcl} a - \ b + 3c = 0 \\ 2b - 7c = 0 \\ 2b - 7c = 0 \end{array} \xrightarrow{-\rho_2+\rho_3} \begin{array}{rcl} a - \ b + 3c = 0 \\ 2b - 7c = 0 \\ 0 = 0 \end{array}$$

We are back to the prior part. The basis is

$$\{ \begin{pmatrix} 1/2 \\ 7/2 \\ 1 \end{pmatrix} \}$$

and the dimension is 1.