Isomorphisms are correspondences between bases.

1. Consider $\mathcal{P}_2$.

a) We could verify that $B = \langle x^2+x, x^2, x^2+1\rangle$ is a basis (but we won't, just in the interests of time). What is the dimension of $\mathcal{P}_2$?

Solution: There are three members of the basis, so V is 3 dimenstional.

b) Represent $3x^2+2x+1$ in terms of that basis. That is, find c_1, c_2, c_3 in this equation.

$$c_1(x^2+x)+c_2(x^2)+c_3(x^2+1) = 3x^2+2x+1$$

Solution: For the values of c_1, c_2, and c_3 in this equation

$$c_1(x^2+x)+c_2(x^2)+c_3(x^2+1) = 3x^2+2x+1$$

we get a linear system, where the first equation finds the coefficient of x^2 on both sides, etc.

$$\begin{aligned} c_1+c_2+c_3 &= 3 \\ c_1 \qquad\quad &= 2 \\ c_3 &= 1 \end{aligned}$$

By eye, we get $c_1 = 2$, $c_3 = 1$, and $c_2 = 0$.

$$\mathrm{Rep}_B(3x^2+2x+1) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

c) Represent ax^2+bx+c in terms of that basis. That is, find c_1, c_2, c_3 in this equation.

$$c_1(x^2+x)+c_2(x^2)+c_3(x^2+1) = ax^2+bx+c$$

Solution: We are given a, b, and c. We want the values of c_1, c_2, and c_3 in this equation.

$$c_1(x^2+x)+c_2(x^2)+c_3(x^2+1) = ax^2+bx+c$$

As in the prior problem, get the linear system.

$$\begin{aligned} c_1+c_2+c_3 &= a \\ c_1 \qquad\quad &= b \\ c_3 &= c \end{aligned}$$

Also as in the prior problem, by eye we get $c_1 = b$, $c_3 = c$, and $c_2 = a - b - c$.

$$b \cdot (x^2 + x) + (a - b - c) \cdot (x^2) + c \cdot (x^2 + 1) = ax^2 + bx + c$$

2. Consider this subspace of $\mathbb{R}^4$.

$$W = \{ \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} \mid p - q + 2s = 0 \}$$

a) Find a basis D for the space W (because of time we won't verify that it is linearly independent). What is the dimension of W?

Solution: We have this.

$$\{ \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} \mid p - q + 2s = 0 \} = \{ \begin{pmatrix} q - 2s \\ q \\ r \\ s \end{pmatrix} \mid q, r, s \in \mathbb{R} \}$$

$$= \{ q \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + r \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \mid q, r, s \in \mathbb{R} \}$$

So we take this to be the basis.

$$D = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

This basis has three elements, so W is 3 dimensional.

3. The dimensions are the same so we can make an isomorphism by making a correspondence between basis elements. $f\colon V \to W$.

a) Associate the first member of B, the polynomial $x^2 + x$, with the second member of D. Associate the polynomial x^2 with the third member of D. Associate $x^2 + 1$ with the first member of D.

Solution:

$$x^2 + x \longleftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad x^2 \longleftrightarrow \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad x^2 + 1 \longleftrightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

b) Extend linearly, that is, give the formula for $f(ax^2 + bx + c)$ by using the first question, and that the function preserves addition and scalar multiplication.

Solution:

$$
\begin{aligned}
f(ax^2 + bx + c) &= f(\ b \cdot (x^2 + x) + (a - b - c) \cdot (x^2) + c \cdot (x^2 + 1)\ ) \\
&= f(\ b \cdot (x^2 + x)\ ) + f(\ (a - b - c) \cdot (x^2)\ ) + f(\ c \cdot (x^2 + 1)\ ) \\
&= b \cdot f(x^2 + x) + (a - b - c) \cdot f(x^2) + c \cdot f(x^2 + 1) \\
&= b \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + (a - b - c) \cdot \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -2a + 2b + 3c \\ c \\ b \\ a - b - c \end{pmatrix}
\end{aligned}
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