1. Consider $\vec{v} \in \mathcal{P}_2$ given by $\vec{v} = x^2 + 2x + 3$.

a) Represent this vector with respect to $B = \langle 1 - x, 1 + x + x^2, x^2 \rangle$.

Solution: To find the coefficients in this equation.

$$x^2 + 2x + 3 = c_1 \cdot (1 - x) + c_2 \cdot (1 + x + x^2) + c_3 \cdot (x^2)$$

we can do Gauss's method on this linear system.

$$\begin{aligned} c_1 + c_2 \phantom{{}+c_3} &= 3 \\ -c_1 + c_2 \phantom{{}+c_3} &= 2 \\ c_2 + c_3 &= 1 \end{aligned} \xrightarrow{\rho_1+\rho_2} \begin{aligned} c_1 + c_2 \phantom{{}+c_3} &= 3 \\ 2c_2 \phantom{{}+c_3} &= 5 \\ c_2 + c_3 &= 1 \end{aligned}$$

$$\xrightarrow{(-1/2)\rho_2+\rho_3} \begin{aligned} c_1 + c_2 \phantom{{}+c_3} &= 3 \\ 2c_2 \phantom{{}+c_3} &= 5 \\ c_3 &= -3/2 \end{aligned}$$

We get this.

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 1/2 \\ 5/2 \\ -3/2 \end{pmatrix}$$

b) Represent it with respect to $D = \langle 1 + x^2, 1 + x, 1 \rangle$.

Solution: From

$$x^2 + 2x + 3 = c_1 \cdot (1 + x^2) + c_2 \cdot (1 + x) + c_3 \cdot (1)$$

by eye the equation of x^2's gives $c_1 = 1$, the equation of x's gives $c_2 = 2$, and then from the equation of constants $c_1 + c_2 + c_3 = 3$ we have $c_3 = 0$.

$$\mathrm{Rep}_D(\vec{v}) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

c) Represent B's basis vectors $\vec{\beta}_1$, $\vec{\beta}_2$, and $\vec{\beta}_3$ with respect to D.

Solution: For $\mathrm{Rep}_D(\vec{\beta}_1)$, solve this.

$$1 - x = c_1 \cdot (1 + x^2) + c_2 \cdot (1 + x) + c_3 \cdot (1)$$

By eye we get $c_1 = 0$, $c_2 = 1$, and $c_3 = 2$.

$$\mathrm{Rep}_D(1 - x) = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

For $\mathrm{Rep}_D(\vec{\beta}_2)$, solve this.

$$1+x+x^2 = c_1\cdot(1+x^2)+c_2\cdot(1+x)+c_3\cdot(1)$$

By eye, $c_1=1$, $c_2=1$, and $c_3=-1$.

$$\mathrm{Rep}_D(1+x+x^2) = \begin{pmatrix}1\\1\\-1\end{pmatrix}$$

Finally, for $\mathrm{Rep}_D(\vec{\beta}_2)$

$$x^2 = c_1\cdot(1+x^2)+c_2\cdot(1+x)+c_3\cdot(1)$$

we get this.

$$\mathrm{Rep}_D(x^2) = \begin{pmatrix}1\\0\\-1\end{pmatrix}$$

d) Find the change of basis matrix $\mathrm{Rep}_{B,D}(\mathrm{id})$

Solution: To find the matrix representing any map, not just the identity, we first determine what the map does to the basis vectors in the domain. In this case we have this.

$$\vec{\beta}_1 \xmapsto{\mathrm{id}} \vec{\beta}_1 \quad \vec{\beta}_2 \xmapsto{\mathrm{id}} \vec{\beta}_2 \quad \vec{\beta}_3 \xmapsto{\mathrm{id}} \vec{\beta}_3$$

Next, represent those outputs with respect to the basis vectors in the codomain, the output space. We did this work in the prior item.

$$\mathrm{Rep}_D(1-x) = \begin{pmatrix}0\\-1\\2\end{pmatrix} \quad \mathrm{Rep}_D(1+x+x^2) = \begin{pmatrix}1\\1\\-1\end{pmatrix} \quad \mathrm{Rep}_D(x^2) = \begin{pmatrix}1\\0\\-1\end{pmatrix}$$

Then the answer puts the columns together.

$$\mathrm{Rep}_{B,D}(\mathrm{id}) = \begin{pmatrix}0 & 1 & 1\\-1 & 1 & 0\\2 & -1 & -1\end{pmatrix}$$

e) Use it to convert $\mathrm{Rep}_B(\vec{v})$ to $\mathrm{Rep}_D(\vec{v})$.

Solution: Do the matrix vector multiplication.

$$\mathrm{Rep}_{B,D}(\mathrm{id})\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 5/2 \\ -3/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \mathrm{Rep}_D(\vec{v})$$

2. Find the change of basis matrix to convert from representations with respect to the first basis to representations with respect to the second.

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \rangle \qquad D = \langle \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rangle$$

Solution: First,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

gives this.

$$\mathrm{Rep}_D(\vec{\beta}_1) = \begin{pmatrix} 1/2 \\ -1 \end{pmatrix}$$

In addition,

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

gives this.

$$\mathrm{Rep}_D(\vec{\beta}_2) = \begin{pmatrix} 1/2 \\ -2 \end{pmatrix}$$

So the change of basis matrix is this.

$$\mathrm{Rep}_{B,D}(\mathrm{id}) = \begin{pmatrix} 1/2 & 1/2 \\ -1 & -2 \end{pmatrix}$$

3. Which of these matrices could be change of basis matrices?

a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Solution: This matrix is nonsingular, for instance since Gauss's method gives an echelon form matrix without any zero rows. So it could be used to change bases.

b) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Solution: This matrix is singular; Gauss's method gives an echelon form matrix with a zero row. So it could not be used as a change of basis matrix.