1. Fix the vector spaces $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$. Consider the map $h\colon V \to W$, which is a homomorphism.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+z \\ y+z \end{pmatrix}$$

a) Find the null space, $\mathscr{N}(h)$.

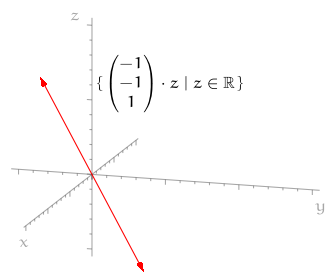
Solution: We have

$$h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} x+z \\ y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which leads to $x+z=0$ and $y+z=0$.

$$\mathscr{N}(h) = \{\begin{pmatrix} -z \\ -z \\ z \end{pmatrix} \mid z \in \mathbb{R}\} = \{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \cdot z \mid z \in \mathbb{R}\}$$

Here is a picture.



b) Give a basis B_N for the null space.

Solution:

$$B_N = \langle \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \rangle$$

c) Find the range space, $\mathscr{R}(h)$

Solution: The range space is all of $W = \mathbb{R}^2$, as we can see by restricting the inputs to $z = 0$.

d) Expand B_N, that is, add vectors to it, to make a basis for the domain, B_V.

Solution: There are infinitely many ways, but here is one. (I made these vectors, just plucked them out of the air, but in such a way that the result is linearly independent.)

$$B_V = \langle \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rangle$$

Checking that it is a basis is striaghtforward. Here is linear independence:

$$c_1 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

leads to

$$\begin{aligned} -c_1 + c_2 + c_3 &= 0 \\ -c_1 \qquad + c_3 &= 0 \\ c_1 + c_2 \qquad &= 0 \end{aligned}$$

and Gauss's method gives that the only solution is the trivial one $c_1 = 0$, $c_2 = 0$, $c_3 = 0$. With that, because the domain $V = \mathbb{R}^3$ is 3-dimensional, this linearly independent set with three members must be a basis.

e) For all the members of B_v that you've added, compute the image when you apply h.

Solution:

$$h(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad h(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

f) Call the sequence of resulting vectors B_R. Check that it is a basis for the range, $W = \mathbb{R}^2$.

Solution: We have this.

$$B_R = \langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$$

It is linearly indepencdent since the second member is not a multiple of the first. Because the range $W = \mathbb{R}^2$ is 2-dimensional, this two-member linearly independent set must be a basis.

g) Conclude that (for this map) the nullity plus the rank equals the dimension of the domain.

Solution: The nullity is the dimension of the null space, 1. The rank is the dimension of the range space, 2. Together they add to the dimension of the domain, since $V = \mathbb{R}^3$ is 3-dimensional/