1. Verify that this map $h\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ is an isomorphism.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b \\ a-b \\ c \\ c+d \end{pmatrix}$$

Solution: We must check (1) and (2) from the definition. Each of these, in turn, involves checking two things.

We first check that the map h is one-to-one. Assume $h(\vec{v}_1) = h(\vec{v}_2)$.

$$h(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) = h(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})$$

$$\begin{pmatrix} a_1+b_1 \\ a_1-b_1 \\ c_1 \\ c_1+d_1 \end{pmatrix} = \begin{pmatrix} a_2+b_2 \\ a_2-b_2 \\ c_2 \\ c_2+d_2 \end{pmatrix} \qquad (*)$$

Comparing third entries gives that $c_1 = c_2$, and then the fourth entries give that $d_1 = d_2$. Comparing the first and second entries gives that $a_1 + b_1 = a_2 + b_2$ and $a_1 - b_1 = a_2 - b_2$. Adding these together gives that $a_1 = a_2$, and thus $b_1 = b_2$. *Remark:* a person can wonder what to do if the relationships are too complicated to eyeball. Equation $(*)$ gives this linear system,

$$\begin{aligned} a_1 + b_1 \qquad\qquad - a_2 - b_2 \qquad\qquad &= 0 \\ a_1 - b_1 \qquad\qquad - a_2 + b_2 \qquad\qquad &= 0 \\ c_1 \qquad\qquad - c_2 \qquad &= 0 \\ c_1 + d_1 \qquad\qquad - c_2 - d_2 &= 0 \end{aligned}$$

and Gauss's method

$$\left(\begin{array}{cccccccc|c} 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 \end{array}\right) \xrightarrow{-1\rho_1+\rho_2} \left(\begin{array}{cccccccc|c} 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{-1\rho_3+\rho_4} \left(\begin{array}{cccccccc|c} 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{array}\right)$$

gives that $d_1 = d_2$, $c_1 = c_2$, $b_1 = b_2$ and $a_1 = a_2$, and so $\vec{v}_1 = \vec{v}_2$.

To finish condition (1), we will show that the function h is onto. For these, often a numerical example is a help. What input does it take to produce this output?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b \\ a-b \\ c \\ c+d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

We need $c = 3$, and $d = 1$. We also need $a = 3/2$ and $b = -1/2$. In general, to get this output

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b \\ a-b \\ c \\ c+d \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

we can use $c = z$ and $d = w - z$, and as well $a = (x+y)/2$ and $b = (x-y)/2$.

Condition (2) is more of a computation. We first show that this map respects addition.

$$\begin{aligned}
h(\vec{v}_1 + \vec{v}_2) &= h(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) \\
&= h(\begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}) \\
&= \begin{pmatrix} (a_1 + a_2) + (b_1 + b_2) \\ (a_1 + a_2) - (b_1 + b_2) \\ c_1 + c_2 \\ (c_1 + c_2) + (d_1 + d_2) \end{pmatrix} \\
&= \begin{pmatrix} (a_1 + b_1) + (a_2 + b_2) \\ (a_1 - b_1) + (a_2 - b_2) \\ c_1 + c_2 \\ (c_1 + d_1) + (c_2 + d_2) \end{pmatrix} \\
&= \begin{pmatrix} a_1 + b_1 \\ a_1 - b_1 \\ c_1 \\ c_1 + d_1 \end{pmatrix} + \begin{pmatrix} a_2 + b_2 \\ a_2 - b_2 \\ c_2 \\ c_2 + d_2 \end{pmatrix} \\
&= h(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) + h(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix})
\end{aligned}$$

For the other half of condition (2), we show that the function h preserves scalar multiplica-

tion.

$$\begin{aligned}
h(r\vec{v}) &= h(r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \\
&= h(\begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}) \\
&= \begin{pmatrix} ra + rb \\ ra - rb \\ rc \\ rc + rd \end{pmatrix} \\
&= r \cdot \begin{pmatrix} a + b \\ a - b \\ c \\ c + d \end{pmatrix} \\
&= r \cdot h(\begin{pmatrix} a & b \\ c & d \end{pmatrix})
\end{aligned}$$

2. Verify condition (2) of the argument showing that this map $f\colon \mathcal{P}_2 \to \mathcal{P}_2$ is an automorphism.

$$ax^2 + bx + c \mapsto (a - c)x^2 + (b - c)x + c$$

Solution: We first show that f preserves addition.

$$\begin{aligned}
f(\vec{v}_1 + \vec{v}_2) &= f(\ (a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2)\ ) \\
&= f(\ (a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2)\ ) \\
&= (a_1 + a_2 - (c_1 + c_2))x^2 + (b_1 + b_2 - (c_1 + c_2))x + (c_1 + c_2) \\
&= \big((a_1 - c_1)x^2 + (b_1 - c_1)x + c\big) + \big((a_2 - c_2)x^2 + (b_2 - c_2)x + c_2\big) \\
&= f(\vec{v}_1) + f(\vec{v}_2)
\end{aligned}$$

We finish by showing that f preserves scalar multiplication.

$$\begin{aligned}
f(r \cdot \vec{v}) &= f(\ r \cdot (ax^2 + bx + c)\ ) \\
&= f(\ (ra)x^2 + (rb)x + (rc)\ ) \\
&= (ra - rc)x^2 + (rb - rc)x + rc \\
&= r \cdot \big((a - c)x^2 + (b - c)x + c\big) \\
&= r \cdot f(\vec{v})
\end{aligned}$$