1. Find the reduced echelon form of this matrix.

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & -3 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & -3 \end{pmatrix} \xrightarrow{(-1/2)\rho_1+\rho_3} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1/2 & -1 \\ 0 & -3/2 & -3 \end{pmatrix}$$

$$\xrightarrow{-3\rho_2+\rho_3} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1/2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[-2\rho_2]{(1/2)\rho_1} \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{(-1/2)\rho_2+\rho_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

2. Show that this matrix is row equivalent to the one in the prior question.

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 2 & 0 & -2 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 2 & 0 & -2 \end{pmatrix} \xrightarrow{(-2/3)\rho_1+\rho_3} \begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 0 & -4/3 & -8/3 \end{pmatrix}$$

$$\xrightarrow{(2/3)\rho_2+\rho_3} \begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[(1/2)\rho_2]{(1/3)\rho_1} \begin{pmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{(-2/3)\rho_2+\rho_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

3. Find what combination of these rows

$$(2\ \ 1\ \ 0) \quad (1\ \ 0\ \ -1) \quad (1\ \ -1\ \ -3)$$

makes the row $(3\ \ 2\ \ 1)$.

Solution: We want to find c_1, c_2, c_3 such that this holds.

$$c_1 \cdot (2\ \ 1\ \ 0) + c_2 \cdot (1\ \ 0\ \ -1) + c_3 \cdot (1\ \ -1\ \ -3) = (3\ \ 2\ \ 1)$$

We get a linear system and solve it.

$$\begin{aligned} 2c_1 + \ \ c_2 + \ \ c_3 &= 3 \\ 1c_1 \qquad\quad - \ \ c_3 &= 2 \\ -c_2 - 3c_3 &= 1 \end{aligned}$$

Gauss's method gives this.

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & -1 & -3 & 1 \end{array}\right) \xrightarrow{(-1/2)\rho_1+\rho_2} \left(\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 0 & -1/2 & -3/2 & 1/2 \\ 0 & -1 & -3 & 1 \end{array}\right) \xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 0 & -1/2 & -3/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

There are infinitely many solutions. One is to take $c_3 = 0$ and get $c_2 = -1$ and $c_1 = 2$.