Fix these matrices.

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 3 & 6 & 1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1. We will perform Gauss-Jordan reduction on the matrix in two ways.

a) Perform the step $-3\rho_1 + \rho_3$ on M to get a matrix A.

Solution:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 3 & 6 & 1 \end{pmatrix} \xrightarrow{-3\rho_1+\rho_3} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b) Instead, multiply M from the left by this matrix.

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

Note that this matrix is the result of starting with I and performing $-3\rho_1 + \rho_3$.

Solution:

$$R_1 M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 3 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A$$

c) Next, perform the step $(1/4)\rho_2$ on A to get a matrix B.

Solution:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{(1/4)\rho_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

d) Instead, multiply A from the left by this matrix.

$$R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that this matrix is the result of starting with I and performing $(1/4)\rho_2$.

Solution:

$$R_2B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = C$$

e) Finally, perform the step $-2\rho_2 + \rho_1$ on B.

Solution:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-2\rho_2+\rho_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

f) Instead, multiply B from the left by this matrix.

$$R_3 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that this matrix is the result of starting with I and performing $-2\rho_2 + \rho_1$.

Solution:

$$R_3C = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

g) So we have $R_3R_2R_1 \cdot M = I$. Multiply R_2R_1, then find $R_3(R_2R_1)$.

Solution: First (note that subscripts come in descending order).

$$R_2R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

and then here is the total.

$$R_3 \cdot (R_2R_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

2. Combine the two matrices M and I by writing the numbers from M, and next to those writing the numbers from I, with a vertical line separating the two.

a) Perform $-3\rho_1 + \rho_3$ on the combination.

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 3 & 6 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-3\rho_1+\rho_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

b) On the result, perform $(1/4)\rho_2$.

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right) \xrightarrow{(1/4)\rho_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

c) Finish by doing $-2\rho_2 + \rho_1$. Compare the right hand side to $R_3R_2R_1$ from the prior question.

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right) \xrightarrow{-2\rho_2+\rho_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$