*When you are working on this material, you must work entirely on your own. You may use the book or your notes. But you may not work with other people, either in the class our outside of it, or use other books or the Internet. If you have any questions, email me.*

**Instructions: Do eight of these ten questions. You must show your work.**

1. Consider this linear system.

$$\begin{aligned} x+3y+z+w&=6\\ -y+2w&=-4\\ 2x+5y+2z+4w&=8\\ x+7w&=-6 \end{aligned}$$

a) Solve it using Gauss' method.

**Solution:**

$$\left(\begin{array}{cccc|c} 1&3&1&1&6\\ 0&-1&0&2&-4\\ 2&5&2&4&8\\ 1&0&0&7&-6 \end{array}\right) \xrightarrow[-1\rho_1+\rho_4]{-2\rho_1+\rho_3} \left(\begin{array}{cccc|c} 1&3&1&1&6\\ 0&-1&0&2&-4\\ 0&-1&0&2&-4\\ 0&-3&-1&6&-12 \end{array}\right)$$

$$\xrightarrow[-3\rho_2+\rho_4]{-1\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1&3&1&1&6\\ 0&-1&0&2&-4\\ 0&0&0&0&0\\ 0&0&-1&0&0 \end{array}\right)$$

$$\xrightarrow{\rho_3\leftrightarrow\rho_4} \left(\begin{array}{cccc|c} 1&3&1&1&6\\ 0&-1&0&2&-4\\ 0&0&-1&0&0\\ 0&0&0&0&0 \end{array}\right)$$

b) State whether there is a no solution, a single solution, or infinitely many solutions.

**Solution:** There are infinitely many solutions.

c) Give the solution set in vector form.

**Solution:** The third equation gives $z=0$. The second is $-y+2w=-4$, and so $y=4+2w$. Substituting into the first equation gives $x+3\cdot(4+2w)+0+w=6$, leading to $x=-6-7w$.

$$\{\begin{pmatrix}-6\\4\\0\\0\end{pmatrix}+\begin{pmatrix}-7\\2\\0\\1\end{pmatrix}\cdot w \mid w\in\mathbb{R}\}$$

2. Represent the linear map with respect to the bases.

$$ax^2+bx+c \xmapsto{f} \begin{pmatrix} a+b \\ b \\ a+c \end{pmatrix} \qquad B=\langle x^2, x^2+x, x^2+1\rangle,\ D=\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \rangle$$

**Solution:** This is the effect of the map on the vectors from B.

$$x^2 \xmapsto{f} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$x^2+x \xmapsto{f} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$x^2+1 \xmapsto{f} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

To represent those with respect to D we fill in the blanks here.

$$\vec{w} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

We get these by eye.

$$\mathrm{Rep}_D(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ 0 \\ 1/3 \end{pmatrix} \qquad \mathrm{Rep}_D(\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ 1/2 \\ 2/3 \end{pmatrix} \qquad \mathrm{Rep}_D(\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}) = \begin{pmatrix} 2 \\ 0 \\ 1/3 \end{pmatrix}$$

Thus this is the matrix.

$$\mathrm{Rep}_{B,D}(f) = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1/2 & 0 \\ 1/3 & 2/3 & 1/3 \end{pmatrix}$$

3. For this matrix

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

a) find the characteristic equation,

**Solution:** $|C - xI| = x^3 - 6x^2$

b) find the eigenvalues.

**Solution:** The eigenvalues are 6 (multiplicity one), and 0 (multiplicity two).

4. For these

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

perform these vector and matrix operations, or state "undefined."

a) $3B - 2A$

**Solution:** Because of a typo, $A = B$. So the answer is this.

$$\begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & -1 \end{pmatrix}$$

b) $BC$

**Solution:** The number of columns in $B$ does not equal the number of rows in $C$, so this is undefined.

c) $CA$

**Solution:** $\begin{pmatrix} 4 & 0 & 2 \\ 9 & 1 & 2 \end{pmatrix}$

d) $CD$

**Solution:** $\begin{pmatrix} 6 \\ 8 \end{pmatrix}$

e) $2BD$

**Solution:** The matrix $2B$ has three columns, while $D$ has two rows. So the product is undefined.

5. For this upper-triangular matrix

$$T = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$$

a) Find the eigenvalues.

**Solution:** $\lambda_1 = 1$, $\lambda_2 = 4$

b) Pick one of the eigenvalues and find an associated eigenvector.

**Solution:** Associated with $\lambda_1 = 1$ is this vector.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Associated with $\lambda_2 = 4$ is this vector.

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

6. Use Gauss's method to find the determinant of this matrix.

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

**Solution:**

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{1\rho_1 + \rho_3} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{\rho_3 \leftrightarrow \rho_4} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Multiplying down the diagonal gives 1, and there were two row swaps, so there were two sign changes. Answer: $+1$.

7. Consider the vector space $\mathcal{M}_{2\times 2}$ with this basis.

$$\vec{\beta}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \vec{\beta}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},\ \vec{\beta}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \vec{\beta}_4 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

Define a map $h\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^3$ by its action on the basis elements.

$$\vec{\beta}_1 \mapsto \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \vec{\beta}_2 \mapsto \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \quad \vec{\beta}_3 \mapsto \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \quad \vec{\beta}_4 \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

a) What is $h(M)$?

$$M = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

**Solution:** We have this.

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

So, $h(M)$ is this.

$$h(\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}) = 0 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ -2 \end{pmatrix}$$

b) Is this map an isomorphism?

**Solution:** No, the dimensions are wrong. We cannot have a dimension four space isomorphic to a dimension three space.

8. Find a basis for this vector space: $\{ax^2 + bx + c \mid 2b + c = 0\}$.

**Solution:** Parameterize.

$$\{ax^2 + (-c/2)x + c \mid a, c \in \mathbb{R}\} = \{a \cdot (x^2) + c \cdot (1 - x/2) \mid a, c \in \mathbb{R}\}$$

We get the sequence $B = \langle x^2, 1 - (x/2)\rangle$.

9. Verify that this map $f\colon \mathcal{P}_2 \to \mathbb{R}^3$ is one-to-one and onto.

$$ax^2 + bx + c \xmapsto{f} \begin{pmatrix} a+b \\ b \\ a+c \end{pmatrix}$$

**Solution:** One way is to represent it with respect to some bases, and then verify that the matrix is nonsingular. Fix the bases $B = \langle x^2, x, 1\rangle$ and $\mathcal{E}_3$. This is the effect of the map on the members of $B$.

$$x^2 \mapsto \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad x \mapsto \begin{pmatrix}1\\1\\0\end{pmatrix} \qquad 1 \mapsto \begin{pmatrix}0\\0\\1\end{pmatrix}$$

They are represented with respect to the codomain basis as here.

$$\mathrm{Rep}_{\mathcal{E}_3}(\begin{pmatrix}1\\0\\1\end{pmatrix}) = \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_3}(\begin{pmatrix}1\\1\\0\end{pmatrix}) = \begin{pmatrix}1\\1\\0\end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_3}(\begin{pmatrix}0\\0\\1\end{pmatrix}) = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

This is the matrix.

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}$$

This is obviously nonsingular since it is already in echelon form.

10. Consider the linear transformation $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ represented with respect to the basis $B$ by the matrix $T$

$$T = \begin{pmatrix}1&0\\2&-1\end{pmatrix} \qquad B = \langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\-2\end{pmatrix}\rangle$$

Find the matrix representing $t$ with respect to this basis.

$$D = \langle \begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\rangle$$

**Solution:** This is the diagram.

$$\begin{array}{ccc} \mathbb{R}^2_{wrt\ B} & \xrightarrow[T]{t} & \mathbb{R}^2_{wrt\ B} \\ {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow \\ \mathbb{R}^2_{wrt\ D} & \xrightarrow[\hat{T}]{t} & \mathbb{R}^2_{wrt\ D} \end{array}$$

For the right side, we find the effect of the identity map on the members of $B$, and then represent the result with respect to $D$. Here is the first

$$\begin{pmatrix}1\\1\end{pmatrix} \xmapsto{id} \begin{pmatrix}1\\1\end{pmatrix} \qquad \mathrm{Rep}_D(\begin{pmatrix}1\\1\end{pmatrix}) = \begin{pmatrix}-1\\3\end{pmatrix}$$

and here is the second.

$$\begin{pmatrix}0\\2\end{pmatrix} \xmapsto{id} \begin{pmatrix}0\\2\end{pmatrix} \qquad \mathrm{Rep}_D(\begin{pmatrix}0\\2\end{pmatrix}) = \begin{pmatrix}-2\\4\end{pmatrix}$$

Thus, the right side of the picture, and it inverse the left side are here.

$$P = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} \qquad P^{-1} = \frac{1}{2}\begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -3/2 & -1/2 \end{pmatrix}$$

The desired matrix $\hat{T}$ is this product.

$$\hat{T} = PTP^{-1} = \begin{pmatrix} -7 & -12 \\ 4 & 7 \end{pmatrix}$$