1. Consider this subset of the collection of all quadratic polynomials, $\mathcal{P}_2$.

$$S = \{px^2 + qx + r \in \mathcal{P}_2 \mid p + q = 0, r \in \mathbb{R}\}$$

a) List three elements and three non-elements.

Solution: Three elements are

$$3 \cdot x^2 - 3 \cdot x + 1 \qquad -4x^2 + 4x + 7 \qquad 0x^2 + 0x + 0$$

Here are three non-elements.

$$3x^2 + 9x + 3 \qquad 25x^2 + 2x - 1 \qquad 17x^2 + 1x + 2$$

b) Show that it is closed under addition (with the standard operation).

Solution: Let $\vec{v}, \vec{w} \in S$. Then

$$\vec{v} = v_2 \cdot x^2 + v_1 \cdot x + v_0 \quad \text{and} \quad \vec{w} = w_2 \cdot x^2 + w_1 \cdot x + w_0$$

such that $v_1 + v_1 = 0$ and $w_2 + w_1 = 0$. Their sum

$$\begin{aligned}\vec{v} + \vec{w} &= (v_2 \cdot x^2 + v_1 \cdot x + v_0) + (w_2 \cdot x^2 + w_1 \cdot x + w_0) \\ &= (v_2 + w_2) \cdot x^2 + (v_1 1 + w_1) \cdot x + (v_0 + w_0)\end{aligned}$$

satisfies that $(v_2 + w_2) + (v_1 + w_1) = (v_2 + v_1) + (w_2 + w_1) = 0 + 0 = 0$. Thus $\vec{v} + \vec{w} \in S$.

c) Show that it is closed under scalar multiplication.

Solution: Fix $\vec{v} \in S$. Then

$$\vec{v} = v_2 \cdot x^2 + v_1 \cdot x + v_0$$

such that $v_1 + v_1 = 0$. Let $r \in \mathbb{R}$. Then

$$r\vec{v} = r \cdot (v_2 x^2 + v_1 x + v_0) = (rv_2)x^2 + (rv_1)x + (rv_0)$$

satisfies that $(rv_2) + (rv_1) = r(v_2 + v_1) = r \cdot 0 = 0$, and thus $r\vec{v} \in S$.

d) Show that addition is commutative, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.

Solution: Let $\vec{v}, \vec{w} \in S$. Then

$$\begin{aligned}\vec{v} + \vec{w} &= (v_2 x^2 + v_1 x + v_0) + (w_2 x^2 + w_1 x + w_0) \\ &= (v_2 + w_2) \cdot x^2 + (v_1 + w_1) \cdot x + (v_0 + w_0)\end{aligned}$$

while

$$\begin{aligned}\vec{w}+\vec{v} &= (w_2x^2+w_1x+w_0)+(v_2x^2+v_1x+v_0)\\ &= (w_2+v_2)\cdot x^2+(w_1+v_1)\cdot x+(w_0+v_0)\end{aligned}$$

and the two are equal, because the coefficents of x^2 are equal, etc. (The coefficients are real numbers, and we know that real number addition commutes. For instance, $v_2+w_2 = w_2+v_2$ because these are sums of real numbers.)

e) Show that addition is associative, $(\vec{v}+\vec{w})+\vec{u} = \vec{v}+(\vec{w}+\vec{u})$.

Solution: Let $\vec{v},\vec{w},\vec{u} \in S$. Then

$$\begin{aligned}(\vec{v}+\vec{w})+\vec{u} &= \big((v_2x^2+v_1x+v_0)+(w_2x^2+w_1x+w_0)\big)+(u_2x^2+u_1x+u_0)\\ &= \big((v_2+w_2)+u_2\big)\cdot x^2+\big((v_1+w_1)+u_1\big)\cdot x+\big((v_0+w_0)+u_0\big)\end{aligned}$$

while

$$\begin{aligned}\vec{v}+(\vec{w}+\vec{u}) &= (v_2x^2+v_1x+v_0)+\big((w_2x^2+w_1x+w_0)+(u_2x^2+u_1x+u_0)\big)\\ &= \big(v_2+(w_2+u_2)\big)\cdot x^2+\big(v_1+(w_1+u_1)\big)\cdot x+\big(v_0+(w_0+u_0)\big)\end{aligned}$$

and the two are equal, because the coefficents of x^2 are equal, etc. (Note that the coefficients are real numbers, and we know that real number addition associates.)

f) Show that there is a zero vector, $\vec{0}$, such that $\vec{v}+\vec{0} = \vec{v}$.

Solution: Consider $0\cdot x^2+0\cdot x+0$. Observe that it is a member of the set S because its coefficient of x^2 plus its coefficient of x totals to 0. Observe also that

$$(0x^2+0x+0)+(v_2x^2+v_1x+v_0) = (0+v_2)x^2+(0+v_1)x+(0+v_0) = v_2x^2+v_1x+v_0$$

so it acts as the identity element with respect to vector addition.

g) Show that each vector has an additive inverse $\vec{w}$ such that $\vec{w}+\vec{v} = \vec{0}$.

Solution: Fix $\vec{v} \in S$; we will produce an additive inverse, a vector that adds to $\vec{v}$ to give the zero element from the prior item. Where $\vec{v} = v_2x^2+v_1x+v_0$, consider the quadratic polynomial whose coefficients are the negatives, $\vec{w} = -v_2x^2-v_1x-v_0$. Obviously the two add to give $\vec{0} = 0x^2+0x+0$.

h) Show that scalar multiplication distributes over scalar addition, $(r+s)\cdot\vec{v} = r\cdot\vec{v}+s\cdot\vec{v}$.

Solution: Fix a vector $\vec{v} \in S$ and two scalars $r,s \in \mathbb{R}$. Then

$$(r+s)\cdot\vec{v} = (r+s)\cdot(v_2x^2+v_1x+v_0) = \big((r+s)v_2\big)\cdot x^2+\big((r+s)v_1\big)\cdot x+\big((r+s)v_0\big)$$

while

$$\begin{aligned} r \cdot \vec{v} + s \cdot \vec{v} &= \big((rv_2)x^2 + (rv_1)x + (rv_0)\big) + \big((sv_2)x^2 + (sv_1)x + (sv_0)\big) \\ &= (rv_2 + sv_2) \cdot x^2 + (rv_1 + sv_1) \cdot x + (rv_0 + sv_0) \end{aligned}$$

and the two are equal quadratic polynomials, since they have the same coefficent of x^2, etc.

i) Show that scalar multiplication distributes over vector addition, $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$.

Solution: Let $r \in \mathbb{R}$ and $\vec{v}, \vec{w} \in S$. Then

$$\begin{aligned} r \cdot (\vec{v} + \vec{w}) &= r \cdot ((v_2x^2 + v_1x + v_0) + (w_2x^2 + w_1x + w_0)) \\ &= r \cdot ((v_2 + w_2)x^2 + (v_1 + w_1)x + (v_0 + w_0)) \\ &= (r \cdot (v_2 + w_2)) \cdot x^2 + (r \cdot (v_1 + w_1)) \cdot x + (r \cdot (v_0 + w_0)) \end{aligned}$$

while

$$\begin{aligned} r \cdot \vec{v} + r \cdot \vec{w} &= r \cdot (v_2x^2 + v_1x + v_0) + r \cdot (w_2x^2 + w_1x + w_0) \\ &= \big((rv_2)x^2 + (rv_1)x + (rv_0)\big) + \big((rw_2)x^2 + (rw_1)x + (rw_0)\big) \\ &= (rv_2 + rw_2) \cdot x^2 + (rv_1 + rw_1) \cdot x + (rv_0 + rw_0) \end{aligned}$$

and the two are equal.

j) Show that ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$.

Solution: Fix scalars $r, s \in \mathbb{R}$ and a vector $\vec{v} = v_2x^2 + v_1x + v_0 \in S$. Then we have this.

$$\begin{aligned} (rs) \cdot \vec{v} = (rs) \cdot (v_2x^2 + v_1x + v_0) &= (rsv_2)x^2 + (rsv_1)x + (rsv_0) \\ &= r \cdot \big((sv_2)x^2 + (sv_1)x + (sv_0)\big) = r \cdot \big(s \cdot (v_2x^2 + v_1x + v_0)\big) \end{aligned}$$

k) Show that multiplication by 1 is the identity operation, $1 \cdot \vec{v} = \vec{v}$.

Solution: Where $\vec{v} \in S$, we have $1 \cdot \vec{v} = 1 \cdot (v_2x^2 + v_1x + v_0) = (1v_2) \cdot x^2 + (1v_1) \cdot x + (1v_0) = v_2x^2 + v_1x + v_0 = \vec{v}$.