We consider the special case of linear transformations, maps where the domain and codomain are equal, $t\colon V \to V$. We further specialize to representations with respect to B, B and D, D.

$$\begin{array}{ccc} V_{wrt\ B} & \xrightarrow[T]{t} & V_{wrt\ B} \\ {\scriptstyle id}\big\downarrow & & {\scriptstyle id}\big\downarrow \\ V_{wrt\ D} & \xrightarrow[\hat{T}]{t} & V_{wrt\ D} \end{array}$$

In matrix terms, we write this.

$$\begin{aligned} \mathrm{Rep}_{D,D}(t) &= \mathrm{Rep}_{B,D}(id) \cdot \mathrm{Rep}_{B,B}(t) \cdot \big(\mathrm{Rep}_{B,D}(id)\big)^{-1} \\ \hat{T} &= P \cdot T \cdot P^{-1} \end{aligned}$$

1. Consider the transformation $t\colon \mathcal{P}_2 \to \mathcal{P}_2$ described by $x^2 \mapsto x+1$, $x \mapsto x^2-1$, and $1 \mapsto 3$.

a) What is $t(3x^2-2x-1)$?

Solution: This basis is particularly easy to work with. Because t is a linear map, we have this.

$$t(3x^2-2x-1) = 3\cdot t(x^2) - 2\cdot t(x) - t(1) = 3\cdot(x-1) - 2\cdot(x^2-1) - (3) = -2x^2+3x-4$$

b) Find $T = \mathrm{Rep}_{B,B}(t)$ where $B = \langle x^2, x, 1\rangle$.

Solution: Because we describe t with the members of B, finding the matrix representation is easy:

$$\mathrm{Rep}_B(t(x^2)) = \begin{pmatrix}0\\1\\1\end{pmatrix}_B \qquad \mathrm{Rep}_B(t(x)) = \begin{pmatrix}1\\0\\-1\end{pmatrix}_B \qquad \mathrm{Rep}_B(t(1)) = \begin{pmatrix}0\\0\\3\end{pmatrix}_B$$

gives this.

$$\mathrm{Rep}_{B,B}(t) \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\1 & -1 & 3\end{pmatrix}$$

c) Find $\hat{T} = \mathrm{Rep}_{D,D}(t)$ where $D = \langle 1, 1+x, 1+x+x^2\rangle$.

Solution: We will find $t(1)$, $t(1+x)$, and $t(1+x+x^2)$, to find how each is represented with respect to D. We are given that $t(1) = 3$, and the other two are easy to see: $t(1+x) = t(1)+t(x) = x^2+2$ and $t(1+x+x^2) = x^2+x+3$. By eye, we get the representation of

each vector

$$\mathrm{Rep}_D(t(1)) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}_D \qquad \mathrm{Rep}_D(t(1+x)) = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}_D \qquad \mathrm{Rep}_D(t(1+x+x^2)) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}_D$$

and thus the representation of the map.

$$\mathrm{Rep}_{D,D}(t) = \begin{pmatrix} 3 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

d) Find the matrix P such that $\hat{T} = PTP^{-1}$.

Solution: From the diagram

$$\begin{tikzcd} V_{wrt\ B} \arrow[r, "t", "T"'] \arrow[d, "id", "P"'] & V_{wrt\ B} \arrow[d, "id", "P"'] \\ V_{wrt\ D} \arrow[r, "t", "\hat{T}"'] & V_{wrt\ D} \end{tikzcd}$$

We can easily calculate $P = \mathrm{Rep}_{B,D}(\mathrm{id})$, and $P^{-1} = \mathrm{Rep}_{D,B}(\mathrm{id})$ for that matter.

$$P = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$