1. Find a basis for the row space of this matrix.

$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 2 & 0 & -1 \\ 5 & 6 & -3 & -4 \end{pmatrix}$$

What is the row rank, the dimension of the row space?

Solution: Do Gauss's method on the matrix.

$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 2 & 0 & -1 \\ 5 & 6 & -3 & -4 \end{pmatrix} \xrightarrow[-5\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -6 & -3 \\ 0 & 6 & -18 & -9 \end{pmatrix} \xrightarrow{-3\rho_2+\rho_3} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -6 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The basis for the row space is $\langle (1\ \ 0\ \ 3\ \ 1), (0\ \ 2\ \ -6\ \ -3) \rangle$. The row rank is 2.

2. For the same matrix, find a basis for the column space. What is the column rank?

Solution: Take the transpose, then apply Gauss's method, and then transpose back.

$$\begin{pmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 3 & 0 & -3 \\ 1 & -1 & -4 \end{pmatrix} \xrightarrow[-1\rho_1+\rho_4]{-3\rho_1+\rho_3} \begin{pmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & -6 & -18 \\ 0 & -3 & -9 \end{pmatrix} \xrightarrow[(3/2)\rho_2+\rho_4]{3\rho_2+\rho_3} \begin{pmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The basis for the column space is this.

$$B = \langle \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} \rangle$$

The column rank is 2

3. What is the rank of the matrix?

Solution: The rank is 2.

4. Consider the vector space V that is the span of this set.

$$S = \{ \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 8 & 0 \\ 6 & 8 \end{pmatrix} \}$$

Find a basis for the span of S. What is the dimension of V?

Solution: Think of the given vectors as "the same" as these, and apply Gauss's method.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 3 & -1 & 1 & 4 \\ 8 & 0 & 6 & 8 \end{pmatrix} \xrightarrow[-8\rho_1+\rho_3]{-3\rho_1+\rho_2} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -4 & -5 & 4 \\ 0 & -8 & -10 & 8 \end{pmatrix} \xrightarrow{-2\rho_2+\rho_3} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -4 & -5 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The desired basis is this.

$$B = \langle \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -4 \\ -5 & 4 \end{pmatrix} \rangle$$

The dimension of V is 2.