Fix the vector spaces $V = \mathbb{R}^2$, and $W = \mathbb{R}^2$, and $X = \mathbb{R}^2$. Fix this vector.

$$\vec{v} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Fix these bases.

$$B_V = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle \quad B_W = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \rangle \quad B_X = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rangle$$

Also fix two linear maps.

$$f(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} a+b \\ 2b \end{pmatrix} \qquad h(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} a-b \\ a+b \end{pmatrix}$$

1. We first represent the two maps with respect to the same pair of bases.

a) Find the matrix representing f: $\mathrm{Rep}_{B_V,B_W}(f)$.

Solution: This is the action of the map on the input basis.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \xmapsto{f} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xmapsto{f} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We represent those results with respect to the output basis.

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} \implies \mathrm{Rep}_{B_W}(\begin{pmatrix} 2 \\ 2 \end{pmatrix}) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} \implies \mathrm{Rep}_{B_W}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

Consequently, this is the matrix representing the map.

$$F = \mathrm{Rep}_{B_V,B_W}(f) = \begin{pmatrix} -2 & 0 \\ 2 & 1/2 \end{pmatrix}$$

b) Find the matrix representing h, that is, $\mathrm{Rep}_{B_V,B_W}(h)$.

Solution: Here is the action of the map on B_V.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \xmapsto{h} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xmapsto{h} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Represent those with respect to the output basis.

$$\begin{pmatrix}0\\2\end{pmatrix} = c_1 \cdot \begin{pmatrix}1\\-1\end{pmatrix} + c_2 \begin{pmatrix}2\\0\end{pmatrix} \quad\Longrightarrow\quad \mathrm{Rep}_{B_W}(\begin{pmatrix}0\\2\end{pmatrix}) = \begin{pmatrix}-2\\1\end{pmatrix}$$
$$\begin{pmatrix}1\\1\end{pmatrix} = c_1 \cdot \begin{pmatrix}1\\-1\end{pmatrix} + c_2 \begin{pmatrix}2\\0\end{pmatrix} \quad\Longrightarrow\quad \mathrm{Rep}_{B_W}(\begin{pmatrix}1\\1\end{pmatrix}) = \begin{pmatrix}-1\\1\end{pmatrix}$$

So this is the matrix representing the map.

$$\mathrm{Rep}_{B_V,B_W}(h) = \begin{pmatrix}-2 & -1\\1 & 1\end{pmatrix}$$

c) Find the column vector representing $\vec{v}$, $\mathrm{Rep}_{B_v}(\vec{v})$.

Solution: Solve this equation.

$$\begin{pmatrix}3\\0\end{pmatrix} = c_1 \begin{pmatrix}1\\1\end{pmatrix} + c_2 \begin{pmatrix}1\\0\end{pmatrix}$$

This is the result.

$$\mathrm{Rep}_{B_V}(\vec{v}) = \begin{pmatrix}0\\3\end{pmatrix}$$

d) Use $\mathrm{Rep}_{B_V,B_W}(f)$ and $\mathrm{Rep}_{B_v}(\vec{v})$ to find $\mathrm{Rep}_{B_W}(f(\vec{v}))$.

Solution: To get the result, find the matrix-vector product.

$$\begin{aligned}\mathrm{Rep}_{B_W}(f(\vec{v})) &= \mathrm{Rep}_{B_v,B_W}(f) \cdot \mathrm{Rep}_{B_V}(\vec{v})\\ &= \begin{pmatrix}2 & 0\\2 & 1/2\end{pmatrix}\begin{pmatrix}0\\3\end{pmatrix}\\ &= \begin{pmatrix}0\\3/2\end{pmatrix}\end{aligned}$$

The point of matrix-vector multiplication is that it does the bookkeeping needed to represent the application of the map to the vector. Restated, the formula for f gives this

$$f(\vec{v}) = f(\begin{pmatrix}3\\0\end{pmatrix}) = \begin{pmatrix}3\\0\end{pmatrix}$$

and from the representation $\mathrm{Rep}_{B_W}(f(\vec{v}))$ we get the same thing.

$$0 \cdot \begin{pmatrix}1\\1\end{pmatrix} + (3/2) \cdot \begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}$$

2. Consider the map $2f\colon V \to W$.

a) Define it.

Solution: Because the function $f\colon V \to W$ is defined this way,

$$f(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} a+b \\ 2b \end{pmatrix}$$

the function $2f$, with the same domain and codomain, is defined this way,

$$2f\,(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} 2a+2b \\ 4b \end{pmatrix}$$

b) Represent it: find $\mathrm{Rep}_{B_V,B_W}(2f)$.

Solution: Because the representation of f that we computed above is this,

$$F = \mathrm{Rep}_{B_V,B_W}(f) = \begin{pmatrix} -2 & 0 \\ 2 & 1/2 \end{pmatrix}$$

the representation of $2f$ given by the results in the book is this.

$$2F = \mathrm{Rep}_{B_V,B_W}(2f) = \begin{pmatrix} -4 & 0 \\ 4 & 1 \end{pmatrix}$$

The point of the defintion of multiplying a matrix by a scalar is that it represents the function that you get by multiplyting by the scalar.

c) Apply it: find $\mathrm{Rep}_{B_W}(2f\,(\vec{v}))$.

Solution: As above, the point of matrix-vector multiplication is that it does the bookkeeping needed to represent the application of the map to the vector.

$$\begin{aligned} \mathrm{Rep}_{B_W}(2f\,(\vec{v})) &= \mathrm{Rep}_{B_v,B_W}(2f) \cdot \mathrm{Rep}_{B_V}(\vec{v}) \\ &= \begin{pmatrix} 4 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 3 \end{pmatrix} \end{aligned}$$

3. Consider the map $f+h\colon V \to W$.

a) Define it.

Solution: Because the function $f\colon V \to W$ is defined this way,

$$f(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} a+b \\ 2b \end{pmatrix}$$

and the function $h\colon V \to W$ is defined this way,

$$h(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} a-b \\ a+b \end{pmatrix}$$

the function $f+h$ is defined this way.

$$f+h\,(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} (a+b)+(a-b) \\ 2b+(a+b) \end{pmatrix} = \begin{pmatrix} 2a \\ a+3b \end{pmatrix}$$

b) Represent it: find $\mathrm{Rep}_{B_V,B_W}(f+h)$.

Solution: The representation of f that we computed above is this.

$$F = \mathrm{Rep}_{B_V,B_W}(f) = \begin{pmatrix} -2 & 0 \\ 2 & 1/2 \end{pmatrix}$$

The representation of h that we computed above is this.

$$H = \mathrm{Rep}_{B_V,B_W}(h) = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$$

The representation of $f+h$ is this.

$$F+H = \mathrm{Rep}_{B_V,B_W}(f+h) = \begin{pmatrix} -2 & 0 \\ 2 & 1/2 \end{pmatrix} + \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & -1 \\ 3 & 3/2 \end{pmatrix}$$

Briefly, we define the '+' operation between two matrices so that $F+H$ represents $f+h$.

c) Apply it: find $\mathrm{Rep}_{B_W}(f+h\,(\vec{v}))$.

Solution: Matrix-vector multiplication is what does the bookkeeping needed to represent the application of the map to the vector.

$$\begin{aligned} \mathrm{Rep}_{B_W}(f+h\,(\vec{v})) &= \mathrm{Rep}_{B_V,B_W}(f+h) \cdot \mathrm{Rep}_{B_V}(\vec{v}) \\ &= \begin{pmatrix} -4 & -1 \\ 3 & 3/2 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 9/2 \end{pmatrix} \end{aligned}$$

4. The bases matter.

a) Find $\mathrm{Rep}_{B_W,B_X}(h)$

Solution: This is the action of the map on B_W.

$$\begin{pmatrix}1\\-1\end{pmatrix} \xmapsto{h} \begin{pmatrix}2\\0\end{pmatrix} \qquad \begin{pmatrix}2\\0\end{pmatrix} \xmapsto{h} \begin{pmatrix}2\\2\end{pmatrix}$$

Next, represent those with respect to the output basis.

$$\begin{pmatrix}2\\0\end{pmatrix} = c_1 \cdot \begin{pmatrix}0\\1\end{pmatrix} + c_2 \begin{pmatrix}2\\2\end{pmatrix} \implies \mathrm{Rep}_{B_X}(\begin{pmatrix}2\\0\end{pmatrix}) = \begin{pmatrix}-2\\1\end{pmatrix}$$
$$\begin{pmatrix}2\\2\end{pmatrix} = c_1 \cdot \begin{pmatrix}0\\1\end{pmatrix} + c_2 \begin{pmatrix}2\\2\end{pmatrix} \implies \mathrm{Rep}_{B_X}(\begin{pmatrix}2\\2\end{pmatrix}) = \begin{pmatrix}0\\1\end{pmatrix}$$

So this is the matrix representing the map.

$$\mathrm{Rep}_{B_W,B_X}(h) = \begin{pmatrix}-2 & 0\\1 & 1\end{pmatrix}$$

b) Contrast it with $\mathrm{Rep}_{B_V,B_W}(h)$.

Solution: Earlier we found that

$$\mathrm{Rep}_{B_V,B_W}(h) = \begin{pmatrix}-2 & -1\\1 & 1\end{pmatrix}$$

so the change in basis results in a change in the matrix representing the map.

5. Consider the map $h \circ f \colon V \to X$.

a) Define it.

Solution: We have this for the action of $h \circ f$.

$$\begin{pmatrix}a\\b\end{pmatrix} \xmapsto{f} \begin{pmatrix}a+b\\2b\end{pmatrix} \xmapsto{h} \begin{pmatrix}(a+b)-(2b)\\(a+b)+(2b)\end{pmatrix} = \begin{pmatrix}a-b\\a+3b\end{pmatrix}$$

b) Represent it: find $\mathrm{Rep}_{B_V,B_X}(h \circ f)$.

Solution: This is the action of $h \circ f$ on the input basis.

$$\begin{pmatrix}1\\1\end{pmatrix} \xmapsto{h} \begin{pmatrix}0\\4\end{pmatrix} \qquad \begin{pmatrix}1\\0\end{pmatrix} \xmapsto{h} \begin{pmatrix}1\\1\end{pmatrix}$$

Next, represent those with respect to the output basis.

$$\begin{pmatrix}0\\4\end{pmatrix} = c_1 \cdot \begin{pmatrix}0\\1\end{pmatrix} + c_2 \begin{pmatrix}2\\2\end{pmatrix} \quad \Longrightarrow \quad \mathrm{Rep}_{B_X}(\begin{pmatrix}0\\4\end{pmatrix}) = \begin{pmatrix}4\\0\end{pmatrix}$$
$$\begin{pmatrix}1\\1\end{pmatrix} = c_1 \cdot \begin{pmatrix}0\\1\end{pmatrix} + c_2 \begin{pmatrix}2\\2\end{pmatrix} \quad \Longrightarrow \quad \mathrm{Rep}_{B_X}(\begin{pmatrix}1\\1\end{pmatrix}) = \begin{pmatrix}0\\1/2\end{pmatrix}$$

So this is the matrix representing the map.

$$\mathrm{Rep}_{B_W,B_X}(h) = \begin{pmatrix}4 & 0\\0 & 1/2\end{pmatrix}$$

Note that this is the same as this matrix-matrix product.

$$\begin{aligned}\mathrm{Rep}_{h\circ f}(B_V, B_X) &= \mathrm{Rep}_h(B_W, B_X) \cdot \mathrm{Rep}_f(B_V, B_W)\\ &= H \cdot F\\ &= \begin{pmatrix}-2 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}-2 & 0\\2 & 1/2\end{pmatrix}\\ &= \begin{pmatrix}4 & 0\\0 & 1/2\end{pmatrix}\end{aligned}$$

c) Apply it: find $\mathrm{Rep}_{B_X}(h \circ f(\vec{v}))$.

Solution: We have this.

$$\mathrm{Rep}_{B_X}(h \circ f(\vec{v})) = \mathrm{Rep}_{B_V,B_X}(h \circ f) \cdot \mathrm{Rep}_{B_V}(\vec{v}) = \begin{pmatrix}4 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}0\\3\end{pmatrix} = \begin{pmatrix}0\\3/2\end{pmatrix}$$