Two.I Vector Space Definition

Linear Algebra, edition four

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Definition and examples

Vector space

1.1 *Definition* A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '$+$' and '$\cdot$' subject to the conditions that for all vectors $\vec{v}, \vec{w}, \vec{u} \in V$, and all *scalars* $r, s \in \mathbb{R}$:

1) the set V is *closed* under vector addition, that is, $\vec{v} + \vec{w} \in V$
2) vector addition is commutative $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
3) vector addition is associative $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
4) there is a *zero vector* $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$
5) each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$

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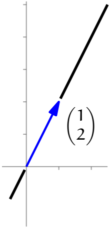
5) each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$
6) the set V is closed under scalar multiplication, that is, $r \cdot \vec{v} \in V$
7) scalar multiplication distributes over addition of scalars $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$
8) scalar multiplication distributes over vector addition $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$
9) ordinary multiplication of scalars associates with scalar multiplication $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$
10) multiplication by the scalar 1 is the identity operation $1 \cdot \vec{v} = \vec{v}$.

Example Let V be the line with slope 2 that passes through the origin in the plane.



$$V = \{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y = 2x \}$$

It is a set consisting of vectors. Here are some of its infinitely many elements.

$$\begin{pmatrix} 4 \\ 8 \end{pmatrix} \quad \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -100 \\ -200 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We will show that this set is a vector space, where the operations are the usual vector addition and scalar multiplication.

We will verify conditions (1)-(10) above.

Before we go through those details, we first reiterate the intuition. A vector space is a place where linear combinations can happen. For instance, this linear combination of vectors from V

$$3\cdot\begin{pmatrix}4\\8\end{pmatrix}+6\cdot\begin{pmatrix}1/2\\1\end{pmatrix}-(1/10)\cdot\begin{pmatrix}-100\\-200\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}5\\10\end{pmatrix} \qquad (*)$$

totals to a member of V, a two-tall vector whose second compenent is twice its first.

To say "linear combinations can happen" requires that we have an addition operation and a scalar multiplication. For an operation to deserve to be called an addition it must satisfy some conditions, such as commutativity, and similarly we have some conditions on scalar multiplication.

But the key conditions, as we illustrated in $(*)$ above, are the closure conditions: when you take a combination of vectors from the space then it must total to another vector from the space.

Now we will verify that V is a vector space under the natural addition and scalar multiplication operations.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

Because this is the first time through the definition we will verify the ten conditions at length.

First is condition (1), closure under addition, that the sum of two members of V is also a member of V.

Take two vectors

$$\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

that are members of V, that is, are subject to the restriction that the second component is twice the first: $y_1 = 2x_1$ and $y_2 = 2x_2$. Their sum

$$\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

is also a member of V because it satisfies the same restriction: $y_1 + y_2 = 2x_1 + 2x_2 = 2(x_1 + x_2)$.

Condition (2), commutativity of addition, is also straightforward. Again we take two vectors

$$\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

subject to $y_1 = 2x_1$ and $y_2 = 2x_2$.

The sums in the two orders are

$$\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \qquad \vec{v}_2 + \vec{v}_1 = \begin{pmatrix} x_2 + x_1 \\ y_2 + y_1 \end{pmatrix}$$

and the two are equal because $x_1 + x_2 = x_2 + x_1$ and $y_1 + y_2 = y_2 + y_1$, as those are sums of real numbers.

Remark Some conditions, including this one, have nothing to do with the $y = 2x$ restriction. We'll say more about this in the next section.

Condition (3), associativity of addition, is like the prior one. The left side is

$$(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \begin{pmatrix} (x_1 + x_2) + x_3 \\ (y_1 + y_2) + y_3 \end{pmatrix}$$

while the right side is

$$\vec{v}_1 + (\vec{v}_2 + \vec{v}_3) = \begin{pmatrix} x_1 + (x_2 + x_3) \\ y_1 + (y_2 + y_3) \end{pmatrix}$$

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Condition (4) asserts that a member of V is an additive identity. We can just exhibit the called-for vector.

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is a member of V since its second component is twice its first. It is the identity element with respect to addition.

$$\vec{v} + \vec{0} = \begin{pmatrix} x + 0 \\ y + 0 \end{pmatrix} = \vec{v}$$

Condition (5), existence of an additive inverse, is also a matter of producing the desired element. Given

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

subject to $y = 2x$, consider this vector.

$$\vec{w} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

Note that $\vec{w} \in V$ because $-y = 2(-x)$ follows from $y = 2x$. Note also that the two add to make the zero vector $\vec{v} + \vec{w} = \vec{0}$.

We finish by verifying the five conditions having to do with scalar multiplication.

Condition (6) is closure under scalar multiplication. Consider a scalar $r \in \mathbb{R}$ and a vector $\vec{v} \in V$

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

(that is, such that $y = 2x$). Then

$$r \cdot \vec{v} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

satisfies the restriction that the second component is twice the first $ry = 2(rx)$, because that equation follows from $y = 2x$. Thus $r\vec{v}$ is also a member of V.

Condition (7) is that real number addition distributes over scalar multiplication. Fix scalars $r, s \in \mathbb{R}$, and a vector $\vec{v} \in V$.

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Then we have this.

$$(r+s) \cdot \vec{v} = \begin{pmatrix} (r+s)x \\ (r+s)y \end{pmatrix} = \begin{pmatrix} rx + sx \\ ry + sy \end{pmatrix} = r\vec{v} + s\vec{v}$$

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For (8), distributivity of vector addition over scalar multiplication, take a scalar $r \in \mathbb{R}$ and two vectors $\vec{v}_1, \vec{v}_2 \in V$.

$$r(\vec{v}_1 + \vec{v}_2) = \begin{pmatrix} r(x_1+x_2) \\ r(y_1+y_2) \end{pmatrix} = \begin{pmatrix} rx_1+rx_2 \\ ry_1+ry_2 \end{pmatrix} = r\vec{v}_1 + r\vec{v}_2$$

For condition (9) suppose $r, s \in \mathbb{R}$ and $\vec{v} \in V$. Compare these two.

$$(rs) \cdot \vec{v} = \begin{pmatrix} (rs)x \\ (rs)y \end{pmatrix} \qquad r \cdot (s\vec{v}) = \begin{pmatrix} r(sx) \\ r(sy) \end{pmatrix}$$

They are equal because the expressions in the components are multiplications of real numbers.

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Condition (10) is simple:

$$1 \cdot \vec{v} = \begin{pmatrix} 1 \cdot x \\ 1 \cdot y \end{pmatrix} = \vec{v}$$

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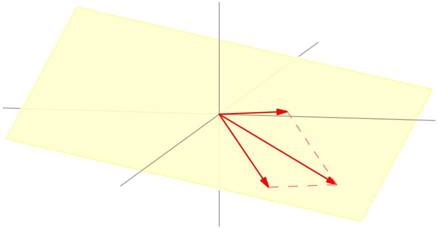
The conclusion: V is a vector space, under the natural addition and scalar multiplication operations.

Example This plane through the origin

$$P = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid 2x + y + 3z = 0 \}$$

is a vector space (under the natural operations). We will verify the two closure conditions (1) and (6); verifying the other conditions is similar to the prior example.

Condition (1) is that two vectors in the plane sum to a vector in the plane. Before we confirm that algebraically, here is the geometry.



For (1), take two members of the plane,

$$\vec{p}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad \vec{p}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

so that both $2x_1 + y_1 + 3z_1 = 0$ and $2x_2 + y_2 + 3z_2 = 0$. The sum is

$$\vec{p}_1 + \vec{p}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

and this sum is in the plane because it satisfies the restriction.

$$2(x_1 + x_2) + (y_1 + y_2) + 3(z_1 + z_2) = (2x_1 + y_1 + 3z_1) + (2x_2 + y_2 + 3z_2) = 0$$

For condition (6) take a member of the plane

$$\vec{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad 2x + y + 3z = 0$$

and multiply by a scalar $r \in \mathbb{R}$.

$$r\vec{p} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$$

Then $r\vec{p}$ is a member of the plane P because $2(rx) + (ry) + 3(rz) = r(2x + y + 3z) = 0$.

$\mathbb{R}^n$

The set of n-tall vectors is a vector space under the natural operations. All ten conditions are easy; we will just verify condition (1). Where

$$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

then the sum

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

is also a member of $\mathbb{R}^n$. (There are no restrictions to check, since every n-tall vector is a member of $\mathbb{R}^n$.)

Example Consider the set of quadratic polynomials.

$$\mathcal{P}_2 = \{\, a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}$$

Some members are $3 + 2x + 1x^2$, $10 + 0x + 5x^2$, and $0 + 0x + 0x^2$.

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Some members are $3 + 2x + 1x^2$, $10 + 0x + 5x^2$, and $0 + 0x + 0x^2$. This is a vector space under the usual operations of polynomial addition

$$(a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

and scalar multiplication.

$$r \cdot (a_0 + a_1 x + a_2 x^2) = (r a_0) + (r a_1)x + (r a_2)x^2$$

Remember the intuition that a vector space is a place where linear combinations can happen. Here is a sample combination in $\mathcal{P}_2$

$$4 \cdot (1 + 2x + 3x^2) - (1/5) \cdot (10 + 5x^2) = 2 + 8x + 11x^2$$

illustrating that a linear combination of quadratic polynomials is a quadratic polynomial.

We won't give a full verification but we will check the closure conditions (1) and (6).

For (1) note that if $\vec{v} = a_0 + a_1 x + a_2 x^2$ and $\vec{w} = (b_0 + b_1 x + b_2 x^2)$ then $\vec{v} + \vec{w} = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$ is also a quadratic polynomial.

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Similarly, for (6) note that if $\vec{v} = a_0 + a_1x + a_2x^2$ then $r \cdot \vec{v} = (ra_0) + (ra_1)x + (ra_2)x^2$ is also a member of $\mathcal{P}_2$.

Example The set of 3×3 matrices

$$\mathcal{M}_{3\times 3} = \{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \mid a_{i,j} \in \mathbb{R} \}$$

is a vector space under the usual matrix addition and scalar multiplication. The check of the ten conditions is straightforward.

Here is a sample linear combination.

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 3 & 1/2 \end{pmatrix} - 3 \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 4 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -5 \\ -1 & -3 & 1 \\ -1 & -9 & -4 \end{pmatrix}$$

The empty set cannot be made a vector space, regardless of which operations we use, because the definition requires that the space contains an additive identity.

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Example The set consisting only of the two-tall zero vector

$$V = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$$

is a vector space (under the usual vector addition and scalar multiplication operations).

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad r \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

1.7 *Definition* A one-element vector space is a *trivial* space.

1.16 *Lemma* In any vector space V, for any $\vec{v} \in V$ and $r \in \mathbb{R}$, we have (1) $0 \cdot \vec{v} = \vec{0}$, (2) $(-1 \cdot \vec{v}) + \vec{v} = \vec{0}$, and (3) $r \cdot \vec{0} = \vec{0}$.

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Proof For (1) note that $\vec{v} = (1+0) \cdot \vec{v} = \vec{v} + (0 \cdot \vec{v})$. Add to both sides the additive inverse of $\vec{v}$, the vector $\vec{w}$ such that $\vec{w} + \vec{v} = \vec{0}$.

$$\begin{aligned}
\vec{w} + \vec{v} &= \vec{w} + \vec{v} + 0 \cdot \vec{v} \\
\vec{0} &= \vec{0} + 0 \cdot \vec{v} \\
\vec{0} &= 0 \cdot \vec{v}
\end{aligned}$$

Item (2) is easy: $(-1 \cdot \vec{v}) + \vec{v} = (-1+1) \cdot \vec{v} = 0 \cdot \vec{v} = \vec{0}$. For (3), $r \cdot \vec{0} = r \cdot (0 \cdot \vec{0}) = (r \cdot 0) \cdot \vec{0} = \vec{0}$ will do. QED

Subspaces and spanning sets

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Example We have seen that in the vector space $\mathbb{R}^2$, the line $y = 2x$

$$S = \{\begin{pmatrix} a \\ 2a \end{pmatrix} \mid a \in \mathbb{R}\} = \{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot a \mid a \in \mathbb{R}\}$$

is a subspace. The operations are the ones from $\mathbb{R}^2$, as required by the definition above. Earlier, to show it is a vector space we checked the ten conditions from the definition. The result below says that to check if something is a subspace there is an easier way.

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Example This subset of $\mathcal{M}_{2\times2}$ is a subspace.

$$S = \{\begin{pmatrix} a & b \\ a & b \end{pmatrix} \mid a, b \in \mathbb{R}\} = \{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot a + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \cdot b \mid a, b \in \mathbb{R}\}$$

Addition and scalar multiplication are the same as in $\mathcal{M}_{2\times2}$.

Example This is not a subspace of $\mathbb{R}^3$.

$$T = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 1 \}$$

It is a subset of $\mathbb{R}^3$ but it is not a vector space. One condition that it violates is that it is not closed under vector addition: here are two elements of T that sum to a vector that is not an element of T.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

(Another reason that it is not a vector space is that it does not satisfy condition (6). Still another is that it does not contain the zero vector.)

2.9 *Lemma* For a nonempty subset S of a vector space, under the inherited operations the following are equivalent statements.

(1) S is a subspace of that vector space

(2) S is closed under linear combinations of pairs of vectors: for any vectors $\vec{s}_1, \vec{s}_2 \in S$ and scalars r_1, r_2 the vector $r_1\vec{s}_1 + r_2\vec{s}_2$ is in S

(3) S is closed under linear combinations of any number of vectors: for any vectors $\vec{s}_1, \ldots, \vec{s}_n \in S$ and scalars $r_1, \ldots, r_n$ the vector $r_1\vec{s}_1 + \cdots + r_n\vec{s}_n$ is an element of S.

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The book has the full proof. For the idea, recall that in the first example of a vector space we remarked that some of the conditions do not depend on the restriction. For instance, there $\vec{v}_1 + \vec{v}_2$ equals $\vec{v}_2 + \vec{v}_1$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} x_2 + x_1 \\ y_2 + y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

because the vector components are real number sums. Only the closure conditions need verification. Statements (2) and (3) above combine the closure conditions into a single one, to make the verification faster.

Example The vector space of quadratic polynomials $\mathcal{P}_2 = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ has a subspace comprised of the linear polynomials $L = \{b_0 + b_1x \mid b_0, b_1 \in \mathbb{R}\}$. By the prior result, to verify that we need only check closure under linear combinations of two members.

$$r \cdot (b_0 + b_1x) + s \cdot (c_0 + c_1x) = (rb_0 + sc_0) + (rb_1 + sc_1)x$$

The right side is a linear polynomial with real coefficients, and so is a member of L. Thus L is a subspace of $\mathcal{P}_2$.

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Example Another subspace of $\mathcal{P}_2$ is the set of quadratic polynomials having three equal coefficients.

$$M = \{a + ax + ax^2 \mid a \in \mathbb{R}\} = \{(1 + x + x^2) \cdot a \mid a \in \mathbb{R}\}$$

Verify that it is a subspace by considering a linear combination of two of its members (under the inherited operations).

$$r \cdot (a + ax + ax^2) + s \cdot (b + bx + bx^2) = (ra + sb) + (ra + sb)x + (ra + sb)x^2$$

The result is a quadratic polynomial with three equal coefficients and so M is closed under linear combinations.

The prior subspace example parametrizes the description.

$$M = \{a + ax + ax^2 \mid a \in \mathbb{R}\} = \{(1 + x + x^2) \cdot a \mid a \in \mathbb{R}\}$$

That proves to be a great way to understand vector spaces.

Example This subset of $\mathbb{R}^3$ is a plane.

$$P = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid 2x - y + z = 0\}$$

We could verify that it is a subspace by checking that it is closed under linear combination, as above. However, that's easier if we first parametrize.

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We could verify that it is a subspace by checking that it is closed under linear combination, as above. However, that's easier if we first parametrize. Solve the one-equation linear system $2x - y + z = 0$ to express the leading variable x in terms of the free variables y and z.

$$P = \{ \begin{pmatrix} (1/2)y - (1/2)z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R}\} = \{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R}\}$$

With the parametrized description

$$P = \{y \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \mid y, z \in \mathbb{R}\} \qquad (*)$$

showing the subspace is closed under linear combinations is straightforward (here, $r_1, r_2 \in \mathbb{R}$).

$$\begin{aligned} r_1 \cdot (y_1 \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + z_1 \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}) + r_2 \cdot (y_2 \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}) \\ = (r_1 y_1 + r_2 y_2) \cdot \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + (r_1 z_1 + r_2 z_2) \cdot \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

Line $(*)$ describes each member of P as a linear combination of the two vectors. Verifying that P is closed then just involves taking a linear combination of linear combinations. Of course, that gives a linear combination.

Example To show that this is a subspace of $\mathcal{M}_{2\times 2}$

$$Q = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a - b = 0 \text{ and } b - c = 0 \}$$

treat that as a two-equation linear system and parametrize. The leading variables are a and b, with free variables c and d.

$$Q = \{ \begin{pmatrix} c & c \\ c & d \end{pmatrix} \mid c, d \in \mathbb{R} \} = \{ c \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid c, d \in \mathbb{R} \}$$

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$$Q=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\mid a-b=0\text{ and }b-c=0\}$$

treat that as a two-equation linear system and parametrize. The leading variables are a and b, with free variables c and d.

$$Q=\{\begin{pmatrix}c&c\\c&d\end{pmatrix}\mid c,d\in\mathbb{R}\}=\{c\begin{pmatrix}1&1\\1&0\end{pmatrix}+d\begin{pmatrix}0&0\\0&1\end{pmatrix}\mid c,d\in\mathbb{R}\}$$

To show Q is a subspace, use the lemma's clause (2) by finding that a linear combination of such matrices is such a matrix.

$$\begin{aligned}r_1\cdot(c_1\begin{pmatrix}1&1\\1&0\end{pmatrix}+d_1\begin{pmatrix}0&0\\0&1\end{pmatrix})+r_2\cdot(c_2\begin{pmatrix}1&1\\1&0\end{pmatrix}+d_2\begin{pmatrix}0&0\\0&1\end{pmatrix})\\=(r_1c_1+r_2c_2)\cdot\begin{pmatrix}1&1\\1&0\end{pmatrix}+(r_1d_1+r_2d_2))\begin{pmatrix}0&0\\0&1\end{pmatrix}\end{aligned}$$

Span

2.13 *Definition* The *span* (or *linear closure*) of a nonempty subset S of a vector space is the set of all linear combinations of vectors from S.

$$[S] = \{c_1\vec{s}_1 + \cdots + c_n\vec{s}_n \mid c_1, \ldots, c_n \in \mathbb{R} \text{ and } \vec{s}_1, \ldots, \vec{s}_n \in S\}$$

The span of the empty subset of a vector space is its trivial subspace.

No notation for the span is completely standard. The square brackets used here are common but so are 'span(S)' and 'sp(S)'.

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The span of the empty subset of a vector space is its trivial subspace.

No notation for the span is completely standard. The square brackets used here are common but so are 'span(S)' and 'sp(S)'.

Example Inside the vector space of all two-wide row vectors, the span of this one-element set

$$S = \{\begin{pmatrix} 1 & 2 \end{pmatrix}\}$$

is this.

$$[S] = \{\begin{pmatrix} a & 2a \end{pmatrix} \mid a \in \mathbb{R}\} = \{\begin{pmatrix} 1 & 2 \end{pmatrix} a \mid a \in \mathbb{R}\}$$

Example This is a subset of $\mathbb{R}^3$.

$$\hat{S} = \{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \}$$

Any vector in the xy-plane is a member of the span $[S]$ because any such vector is a combination of the two; for instance, this system has a solution

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c_2$$

(the top two rows gives a linear system with a unique solution).

Example This is a subset of $\mathbb{R}^3$.

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(the top two rows gives a linear system with a unique solution). But vectors not in the xy-plane are not in the span. For instance, this system does not have a solution.

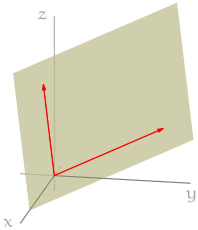
$$\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c_2$$

Example Here is another subset of $\mathbb{R}^3$.

$$P = \{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \}$$

Vectors in the span $[P]$ are combinations of the two, shown in red. The span is a plane through the origin.

$$[P] = \{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot s + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \cdot t \mid s, t \in \mathbb{R} \}$$



2.15 *Lemma* In a vector space, the span of any subset is a subspace.

2.15 *Lemma* In a vector space, the span of any subset is a subspace.

Proof If the subset S is empty then by definition its span is the trivial subspace. If S is not empty then by Lemma 2.9 we need only check that the span $[S]$ is closed under linear combinations of pairs of elements. For a pair of vectors from that span, $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ and $\vec{w} = c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m$, a linear combination

$$\begin{aligned} p\cdot(c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) &+ r\cdot(c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m) \\ &= pc_1\vec{s}_1 + \cdots + pc_n\vec{s}_n + rc_{n+1}\vec{s}_{n+1} + \cdots + rc_m\vec{s}_m \end{aligned}$$

is a linear combination of elements of S and so is an element of $[S]$ (possibly some of the $\vec{s}_i$'s from $\vec{v}$ equal some of the $\vec{s}_j$'s from $\vec{w}$ but that does not matter). QED

Example This illustrates that a span is closed under linear combinations. Where

$$\hat{S} = \{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \} \subset \mathbb{R}^3$$

these are two elements of the span $[\hat{S}]$.

$$\vec{v}_1 = 5 \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = -2 \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 10 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The linear combination of those two $-3\vec{v}_1 + 7\vec{v}_2$ makes another element of the span.

$$-3 \cdot (5 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}) + 7 \cdot (-2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 10 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}) = -29 \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 61 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

This is just an instance of the Linear Combination Lemma: a linear combination of linear combinations is a linear combination.

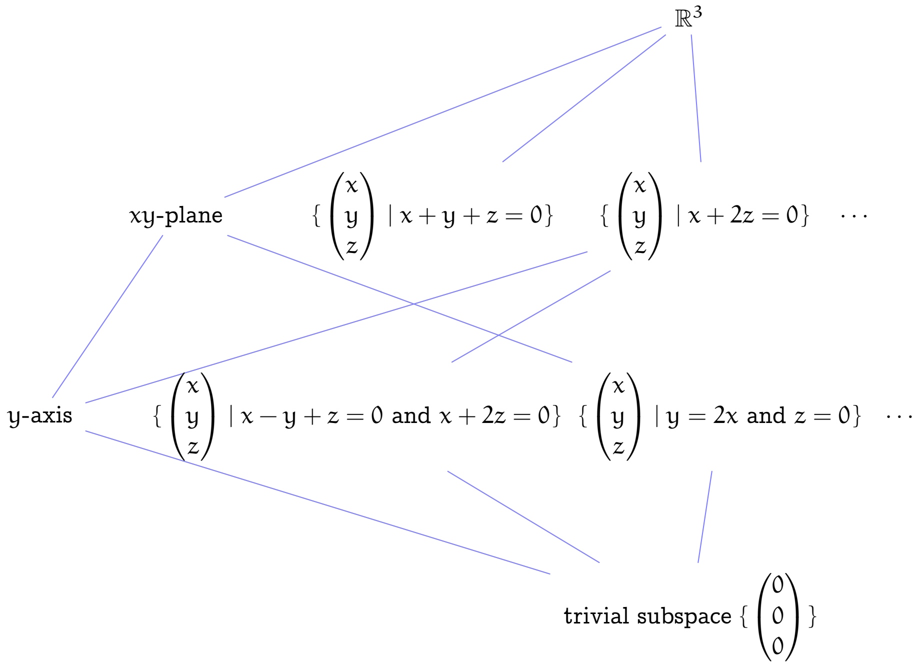
Spaces and their subspaces

Subspaces of $\mathbb{R}^3$

The next slide shows a sample of subspaces of the vector space $\mathbb{R}^3$: the entire space, planes, lines, and the trivial subspace. Subsets are drawn connected to supersets, on the levels directly above and below.

On the level one up from the bottom, the subspaces are lines (in the second and third case, because the conjunction of the two conditions means that each is the intersection of two planes). That is, they are one-dimensional.

On the next level up, the subspaces are planes, two-dimensional. On the top level, the space is 3-space.


$\mathbb{R}^3$
xy-plane
$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 0\}$
$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + 2z = 0\}$
$\cdots$
y-axis
$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x - y + z = 0 \text{ and } x + 2z = 0\}$
$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid y = 2x \text{ and } z = 0\}$
$\cdots$
trivial subspace $\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\}$

Express subspaces as spans

Example This is a subset of $\mathbb{R}^3$.

$$P = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 0 \}$$

Here are three members of this set, and an equation showing that the third is a linear combination of the first two, illustrating that it is a vector space.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -2 \\ -5 \\ 7 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \qquad 3\vec{v}_1 + \vec{v}_2 = \vec{v}_3$$

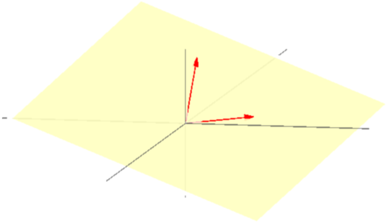
To get a more useful description, take the condition $x + y + z = 0$ to be a one-equation linear system and parametrize.

$$P = \{ \begin{pmatrix} -y - z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R} \} = \{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R} \}$$

Thus the plane is the span of those two vectors.

The subspace $P \subseteq \mathbb{R}^3$ is a plane passing through the origin.

$$P = [\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}]$$



The two vectors from the spanning set

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

are in red. For each, its entire body lies in the plane.

Example For the plane

$$\hat{P} = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + 2z = 0 \}$$

repeat the process

$$\hat{P} = \{ \begin{pmatrix} -2z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R} \} = \{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R} \}$$

to express it as a span.

$$\hat{P} = [\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \}]$$

Example For the plane

$$\hat{P} = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + 2z = 0 \}$$

repeat the process

$$\hat{P} = \{ \begin{pmatrix} -2z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R} \} = \{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R} \}$$

to express it as a span.

$$\hat{P} = [\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \}]$$

Example The xy-plane is a span in a natural way.

$$xy \text{ plane} = [\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}]$$

Example Next we parametrize the lines. The conditions in the set description

$$L = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x - y + z = 0 \text{ and } x + 2z = 0 \}$$

make a linear system.

$$\begin{array}{rcl} x - y + z = 0 & \xrightarrow{-\rho_1+\rho_2} & x - y + z = 0 \\ x + 2z = 0 & & y + z = 0 \end{array}$$

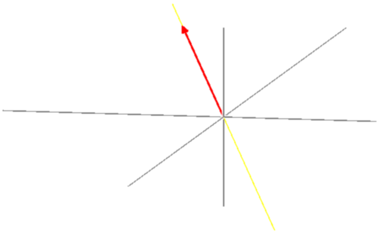
Parametrizing

$$L = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid y = -z \text{ and } x = -2z \}$$

gives this.

$$L = \{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} z \mid z \in \mathbb{R} \} = [\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \}]$$

Here the line, the subspace, is in yellow. In red is the vector used in the description.



In red is the vector used in the description.

$$\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

Its endpoint lies behind the plane of the screen, in the octant where x and y are negative and z is positive.

Example The line

$$\hat{L} = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid y = 2x \text{ and } z = 0 \}$$

is easy to describe in a parametrized way.

$$\begin{aligned} \hat{L} &= \{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y \mid y \in \mathbb{R} \} \\ &= [\{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \}] \end{aligned}$$

Example The line

$$\hat{L} = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid y = 2x \text{ and } z = 0 \}$$

is easy to describe in a parametrized way.

$$\begin{aligned} \hat{L} &= \{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y \mid y \in \mathbb{R} \} \\ &= [\{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \}] \end{aligned}$$

Example The y-axis is also easy to describe as a span.

$$y\text{-axis} = [\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}]$$

Example We can describe the entire space as a span.

$$\mathbb{R}^3 = [\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}]$$

Example We can describe the entire space as a span.

$$\mathbb{R}^3 = [\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}]$$

Example We can do the same for the trivial subspace.

$$\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \} = [\{ \}]$$

(Remember that a sum of zero-many vectors is the zero vector.)

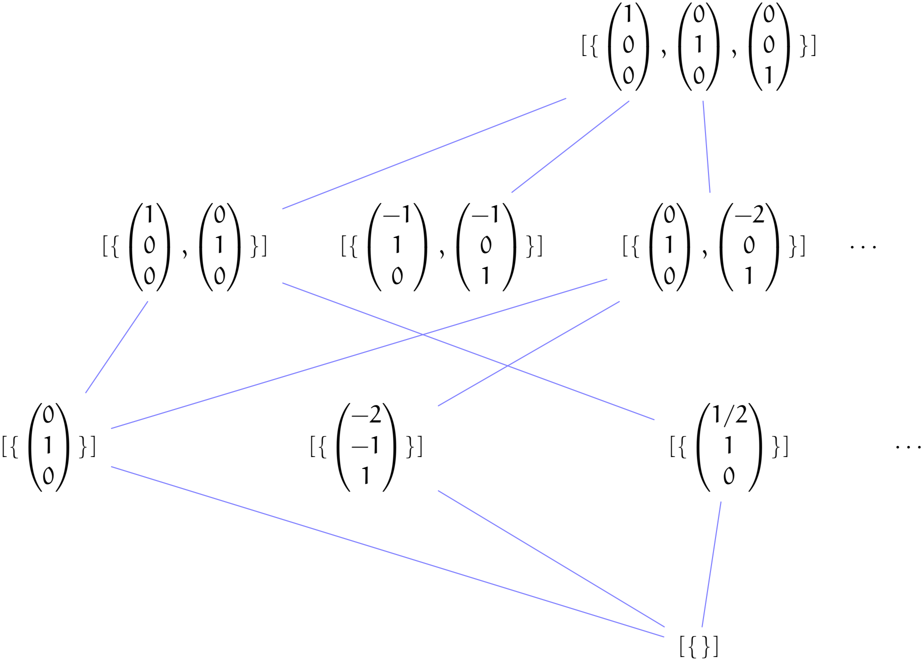
$\mathbb{R}^3$'s diagram reprised

The following slide repeats the diagram of $\mathbb{R}^3$'s subspaces, showing the same subspaces. On this diagram the same subspaces they are described as spans, where the spanning set uses a minimal number of vectors.

By 'minimal' we mean: while we could describe the xy-plane in either of these ways,

$$xy\text{-plane} = [\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}] = [\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-1\\0\end{pmatrix}\}]$$

the second has an extra vector, so the next slide doesn't use that description. (We will soon make this precise.)



$[\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\}]$
$[\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\}]$
$[\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\}]$
$[\{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}-2\\0\\1\end{pmatrix}\}]$
$\cdots$
$[\{\begin{pmatrix}0\\1\\0\end{pmatrix}\}]$
$[\{\begin{pmatrix}-2\\-1\\1\end{pmatrix}\}]$
$[\{\begin{pmatrix}1/2\\1\\0\end{pmatrix}\}]$
$\cdots$
$[\{\}]$

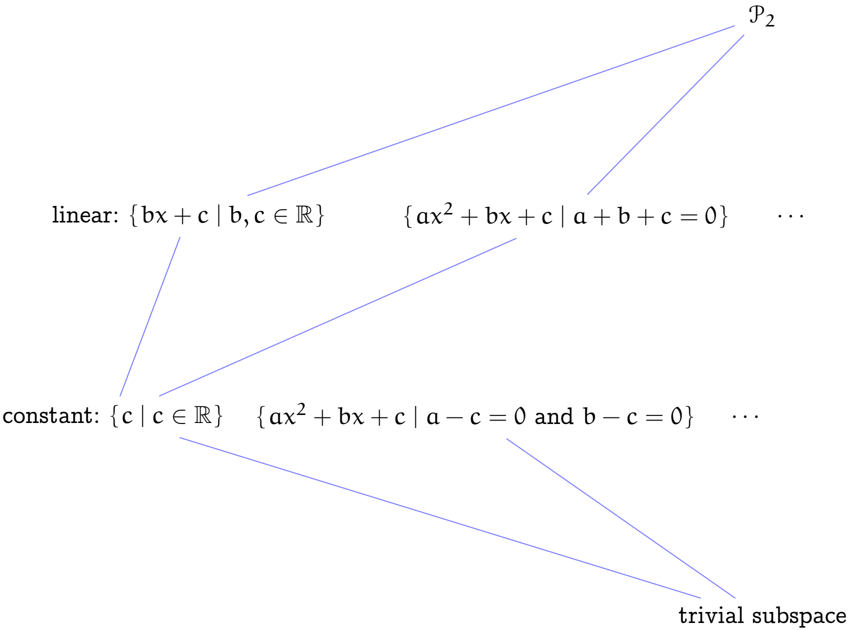
$\mathcal{P}_2$'s subspaces

The next slide has a picture of some of the subspaces of the space of quadratic polynomials. As with the $\mathbb{R}^3$ diagram, subsets are shown connected to supersets on the adjacent level.

With $\mathbb{R}^3$ the geometry gave us a good start for a natural classification of subspaces into planes, lines, etc. In this space there is less of that sense. But there are a couple of natural subspaces.

$$\begin{aligned} \text{linear polynomials} &= \{0x^2 + bx + c \mid b, c \in \mathbb{R}\} \\ \text{constant polynomials} &= \{0x^2 + 0x + c \mid c \in \mathbb{R}\} \end{aligned}$$

These are on the next slide, along with a couple of more-generic spaces.



$\mathcal{P}_2$
linear: $\{bx + c \mid b, c \in \mathbb{R}\}$
$\{ax^2 + bx + c \mid a + b + c = 0\}$
$\cdots$
constant: $\{c \mid c \in \mathbb{R}\}$
$\{ax^2 + bx + c \mid a - c = 0 \text{ and } b - c = 0\}$
$\cdots$
trivial subspace

Parametrizing

Example For the top level, this will do.

$$\mathcal{P}_2 = [\{x^2, x, 1\}] = [\{x^2 + 0x + 0, 0x^2 + x + 0, 0x^2 + 0x + 1\}]$$

Example On the bottom level, the trivial subspace is the span of the empty set.

$$\{0\} = \{0x^2 + 0x + 0\} = [\{\,\}]$$

Parametrizing

Example For the top level, this will do.

$$\mathcal{P}_2 = [\{x^2, x, 1\}] = [\{x^2 + 0x + 0, 0x^2 + x + 0, 0x^2 + 0x + 1\}]$$

Example On the bottom level, the trivial subspace is the span of the empty set.

$$\{0\} = \{0x^2 + 0x + 0\} = [\{\,\}]$$

Example The set of linear polynomials has a natural expression as a span.

$$\{bx + c \mid b, c \in \mathbb{R}\} = [\{x, 1\}]$$

Example The set of constant polynomials is similar.

$$\{0x^2 + 0x + c \mid c \in \mathbb{R}\} = [\{1\}]$$

Example Parametrize $P = \{ax^2 + bx + c \mid a + b + c = 0\}$ in much the same way as for subspaces of real three-space: treat the restriction as a one-equation linear system and parametrize.

$$\begin{aligned} P &= \{ax^2 + bx + c \mid a = -b - c\} \\ &= \{(-b - c) \cdot x^2 + bx + c \mid b, c \in \mathbb{R}\} \\ &= \{(-x^2 + x) \cdot b + (-x^2 + 1) \cdot c \mid b, c \in \mathbb{R}\} \end{aligned}$$

The spanning set is $\{-x^2 + x, -x^2 + 1\}$.

Example Parametrize $P = \{ax^2 + bx + c \mid a + b + c = 0\}$ in much the same way as for subspaces of real three-space: treat the restriction as a one-equation linear system and parametrize.

$$\begin{aligned} P &= \{ax^2 + bx + c \mid a = -b - c\} \\ &= \{(-b - c) \cdot x^2 + bx + c \mid b, c \in \mathbb{R}\} \\ &= \{(-x^2 + x) \cdot b + (-x^2 + 1) \cdot c \mid b, c \in \mathbb{R}\} \end{aligned}$$

The spanning set is $\{-x^2 + x, -x^2 + 1\}$.

Example Simliarly for $\hat{P} = \{ax^2 + bx + c \mid a - c = 0 \text{ and } b - c = 0\}$.

$$\begin{aligned} \hat{P} &= \{ax^2 + bx + c \mid a = c \text{ and } b = c\} \\ &= \{cx^2 + cx + c \mid c \in \mathbb{R}\} \\ &= \{(x^2 + x + 1) \cdot c \mid c \in \mathbb{R}\} \end{aligned}$$

The spanning set is $\{x^2 + x + 1\}$.

Example Parametrize $P=\{ax^2+bx+c \mid a+b+c=0\}$ in much the same way as for subspaces of real three-space: treat the restriction as a one-equation linear system and parametrize.

$$\begin{aligned} P &= \{ax^2+bx+c \mid a=-b-c\} \\ &= \{(-b-c)\cdot x^2+bx+c \mid b,c\in\mathbb{R}\} \\ &= \{(-x^2+x)\cdot b+(-x^2+1)\cdot c \mid b,c\in\mathbb{R}\} \end{aligned}$$

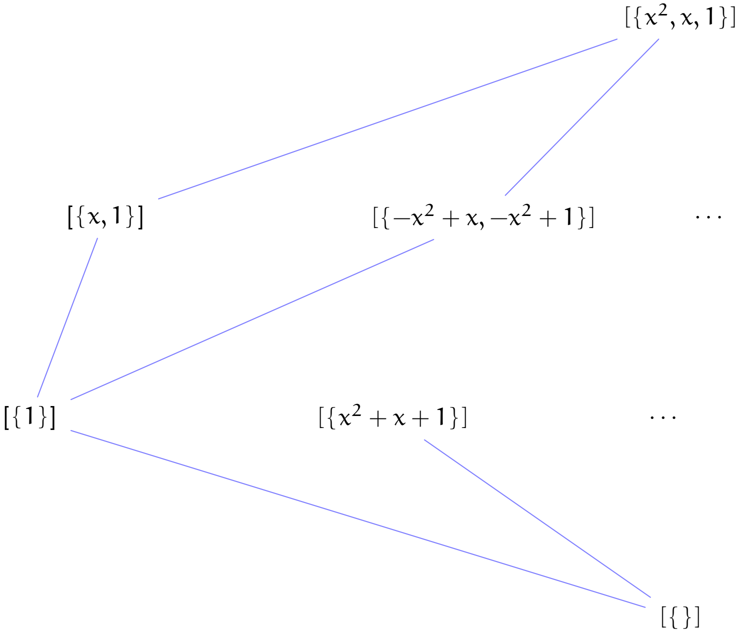
The spanning set is $\{-x^2+x,-x^2+1\}$.

Example Simliarly for $\hat{P}=\{ax^2+bx+c \mid a-c=0 \text{ and } b-c=0\}$.

$$\begin{aligned} \hat{P} &= \{ax^2+bx+c \mid a=c \text{ and } b=c\} \\ &= \{cx^2+cx+c \mid c\in\mathbb{R}\} \\ &= \{(x^2+x+1)\cdot c \mid c\in\mathbb{R}\} \end{aligned}$$

The spanning set is $\{x^2+x+1\}$.

The next slide reprises $\mathcal{P}_2$'s diagram, with the subspaces described as spans.


$[\{x^2, x, 1\}]$
$[\{x, 1\}]$
$[\{-x^2 + x, -x^2 + 1\}]$
$\cdots$
$[\{1\}]$
$[\{x^2 + x + 1\}]$
$\cdots$
$[\{\}]$

Summary

Subspaces are naturally described as spans. In both examples these spans fall naturally into levels, according to the number of elements in a minimal spanning set.

The book's next section gives a precise definition of when a spanning set is 'minimal'. The section after that shows that for any space, two minimal spanning sets have the same number of vectors.