Three.III Computing Linear Maps

Linear Algebra

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Representing Linear Maps with Matrices

Linear maps are determined by the action on a basis

Fix a domain space V with basis $\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$, and a codomain space W. This equation

$$h(\vec{v}) = h(c_1\cdot\vec{\beta}_1+\cdots+c_n\cdot\vec{\beta}_n) = c_1\cdot h(\vec{\beta}_1)+\cdots+c_n\cdot h(\vec{\beta}_n) \qquad (*)$$

says that once we specify where a homomorphism $h\colon V\to W$ sends the basis elements then we have specified where it sends any input vector. We've called this *extending linearly* the action of the map from the basis to the entire domain. We now introduce a scheme for these calculations.

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Example Let $V = \mathcal{P}_2$ and $W = \mathbb{R}^2$, with these bases.

$$B_V = \langle 1, 1+x, 1+x+x^2 \rangle \qquad B_W = \langle \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rangle$$

Let $h\colon \mathcal{P}_2 \to \mathbb{R}^2$ have this action on the domain basis.

$$h(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad h(1+x) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad h(1+x+x^2) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

With this effect on the domain's basis,

$$h(\vec{\beta}_1)=\begin{pmatrix}0\\1\end{pmatrix}\quad h(\vec{\beta}_2)=\begin{pmatrix}3\\2\end{pmatrix}\quad h(\vec{\beta}_3)=\begin{pmatrix}-2\\-1\end{pmatrix}$$

next find the representation of the three outputs with respect to the codomain's basis B_W.

$$\mathrm{Rep}_{B_W}(h(\vec{\beta}_1))=\begin{pmatrix}1/2\\1\end{pmatrix}\qquad\text{since }\begin{pmatrix}0\\1\end{pmatrix}=(1/2)\cdot\begin{pmatrix}2\\0\end{pmatrix}+1\cdot\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\mathrm{Rep}_{B_W}(h(\vec{\beta}_2))=\begin{pmatrix}5/2\\2\end{pmatrix}\qquad\text{since }\begin{pmatrix}3\\2\end{pmatrix}=(5/2)\cdot\begin{pmatrix}2\\0\end{pmatrix}+2\cdot\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\mathrm{Rep}_{B_W}(h(\vec{\beta}_3))=\begin{pmatrix}-3/2\\-1\end{pmatrix}\qquad\text{since }\begin{pmatrix}-2\\-1\end{pmatrix}=(-3/2)\cdot\begin{pmatrix}2\\0\end{pmatrix}-1\cdot\begin{pmatrix}-1\\1\end{pmatrix}$$

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$$\mathrm{Rep}_{B_W}(h(\vec{\beta}_3)) = \begin{pmatrix} -3/2 \\ -1 \end{pmatrix} \qquad \text{since } \begin{pmatrix} -2 \\ -1 \end{pmatrix} = (-3/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Summarize by concatenating those into the matrix representation of h with respect to B_V, B_W.

$$\begin{pmatrix} 1/2 & 5/2 & -3/2 \\ 1 & 2 & -1 \end{pmatrix}$$

The point is that if we represent a domain vector with respect to the domain's basis $\vec{v} = c_1 \cdot \vec{\beta}_1 + c_2 \cdot \vec{\beta}_2 + c_3 \cdot \vec{\beta}_3$ and apply equation $(*)$

$$\begin{aligned} h(c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + c_3\vec{\beta}_3) &= c_1 \cdot h(\vec{\beta}_1) + c_2 \cdot h(\vec{\beta}_2) + c_3 \cdot h(\vec{\beta}_3) \qquad (*) \\ &= c_1 \cdot \left((1/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) \\ &\quad + c_2 \cdot \left((5/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) \\ &\quad + c_3 \cdot \left((-3/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) \end{aligned}$$

then regrouping

$$= ((1/2)c_1 + (5/2)c_2 - (3/2)c_3) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + (1c_1 + 2c_2 - 1c_3) \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

gives the column vector that represents $h(\vec{v})$ with respect to the codomain's basis.

$$\mathrm{Rep}_{B_W}(h(\vec{v})) = \begin{pmatrix} (1/2)c_1 + (5/2)c_2 - (3/2)c_3 \\ 1c_1 + 2c_2 - 1c_3 \end{pmatrix}$$

Thus, to represent an application of the linear map

$$h(\vec{v})$$

we write the representation of the map next to the representation of the input

$$\begin{pmatrix} 1/2 & 5/2 & -3/2 \\ 1 & 2 & -1 \end{pmatrix}_{B_V,B_W} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}_{B_V}$$

and compute the representation of the output.

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$$\mathrm{Rep}_{B_W}(h(\vec{v})) = \begin{pmatrix} (1/2)c_1 + (5/2)c_2 - (3/2)c_3 \\ 1c_1 + 2c_2 - 1c_3 \end{pmatrix}_{B_W}$$

The way the numbers combine to do this computation is: take the dot product of the rows of the matrix with the column representing the input, to get the column vector representing the output.

Matrix representation of a linear map

1.2 *Definition* Suppose that V and W are vector spaces of dimensions n and m with bases B and D, and that $h\colon V \to W$ is a linear map. If

$$\mathrm{Rep}_D(h(\vec{\beta}_1)) = \begin{pmatrix} h_{1,1} \\ h_{2,1} \\ \vdots \\ h_{m,1} \end{pmatrix}_D \quad \ldots \quad \mathrm{Rep}_D(h(\vec{\beta}_n)) = \begin{pmatrix} h_{1,n} \\ h_{2,n} \\ \vdots \\ h_{m,n} \end{pmatrix}_D$$

then

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \ldots & h_{m,n} \end{pmatrix}_{B,D}$$

is the *matrix representation of* h *with respect to* B, D.

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then

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \ldots & h_{m,n} \end{pmatrix}_{B,D}$$

is the *matrix representation of h with respect to B, D*.

(We often omit the subscript B, D.)

Example Consider projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2$ onto the x-axis.

$$\begin{pmatrix} a \\ b \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} a \\ 0 \end{pmatrix}$$

Let these be the domain and codomain bases.

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rangle \qquad D = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rangle$$

Apply the map to each element of the domain basis, and represent the result with respect to the codomain basis

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{so} \quad \mathrm{Rep}_D(\pi(\vec{\beta}_1)) = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$$
$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{so} \quad \mathrm{Rep}_D(\pi(\vec{\beta}_2)) = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

to get the matrix representation of the map.

$$\mathrm{Rep}_{B,D}(\pi) = \begin{pmatrix} -1 & 1 \\ 1/2 & -1/2 \end{pmatrix}$$

Example Again consider projection onto the x-axis

$$\begin{pmatrix} a \\ b \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} a \\ 0 \end{pmatrix}$$

but this time take the input and output bases to be the standard.

$$B = D = \mathcal{E}_2 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$$

Remembering that with respect to the standard basis each vector represents itself,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{so } \mathrm{Rep}_D(\pi(\vec{\beta}_1)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{so } \mathrm{Rep}_D(\pi(\vec{\beta}_2)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

this is $\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(\pi)$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Example We will represent $h\colon \mathbb{R}^2 \to \mathbb{R}$

$$\begin{pmatrix} a \\ b \end{pmatrix} \overset{h}{\longmapsto} 2a + 3b$$

with respect to $\mathcal{E}_2$ and $\mathcal{E}_1$. First find the effect of h on the domain's basis.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto 2 \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto 3$$

Represent those outputs with respect to the codomain's basis.

$$\mathrm{Rep}_{\mathcal{E}_1}(h(\vec{e}_1)) = \begin{pmatrix} 2 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_1}(h(\vec{e}_2)) = \begin{pmatrix} 3 \end{pmatrix}$$

(Those column vectors are 1-tall.) This is 1×2 matrix representation.

$$H = \mathrm{Rep}_{\mathcal{E}_2, \mathcal{E}_1}(h) = \begin{pmatrix} 2 & 3 \end{pmatrix}$$

1.5 *Theorem* Assume that V and W are vector spaces of dimensions n and m with bases B and D, and that $h\colon V \to W$ is a linear map. If h is represented by

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \ldots & h_{m,n} \end{pmatrix}_{B,D}$$

and $\vec{v} \in V$ is represented by

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_B$$

then the representation of the image of $\vec{v}$ is this.

$$\mathrm{Rep}_D(\,h(\vec{v})\,) = \begin{pmatrix} h_{1,1}c_1 + h_{1,2}c_2 + \cdots + h_{1,n}c_n \\ h_{2,1}c_1 + h_{2,2}c_2 + \cdots + h_{2,n}c_n \\ \vdots \\ h_{m,1}c_1 + h_{m,2}c_2 + \cdots + h_{m,n}c_n \end{pmatrix}_D$$

Proof This formalizes the example that began this subsection. See Exercise 33 . QED

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1.6 *Definition* The *matrix-vector product* of a $m\times n$ matrix and a $n\times 1$ vector is this.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ & \vdots & & \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_{1,1}c_1 + \cdots + a_{1,n}c_n \\ a_{2,1}c_1 + \cdots + a_{2,n}c_n \\ \vdots \\ a_{m,1}c_1 + \cdots + a_{m,n}c_n \end{pmatrix}$$

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Example We can perform the matrix-vector product operation without reference to maps, or spaces and bases.

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \cdot 4 + 1 \cdot (-1) + 2 \cdot (-3) \\ 0 \cdot 4 - 2 \cdot (-1) + 5 \cdot (-3) \end{pmatrix} = \begin{pmatrix} 5 \\ -13 \end{pmatrix}$$

Example The product of these two is not defined.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

Example Recall the matrix representing projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2$ onto the x-axis

$$\mathrm{Rep}_{B,D}(\pi) = \begin{pmatrix} -1 & 1 \\ 1/2 & -1/2 \end{pmatrix}$$

with respect to these.

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rangle \quad D = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rangle$$

This domain vector

$$\vec{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

has this representation with respect to the domain basis.

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \text{since } \begin{pmatrix} -1 \\ 5 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The matrix-vector product gives the representation of $\pi(\vec{v})$.

$$\begin{pmatrix} -1 & 1 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

We can check that the prior computation is right. The projection of the vector

$$\pi(\begin{pmatrix}-1\\5\end{pmatrix}) = \begin{pmatrix}-1\\0\end{pmatrix}$$

is represented with respect to the codomain basis D as here.

$$\begin{pmatrix}-1\\0\end{pmatrix} = 1 \cdot \begin{pmatrix}0\\1\end{pmatrix} - (1/2) \cdot \begin{pmatrix}2\\2\end{pmatrix}$$

Example Recall also that the map $h\colon \mathbb{R}^2 \to \mathbb{R}$ with this action

$$\begin{pmatrix} a \\ b \end{pmatrix} \stackrel{h}{\longmapsto} 2a + 3b$$

is represented with respect to the standard bases $\mathcal{E}_2, \mathcal{E}_1$ by a 1×2 matrix.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_1}(h) = \begin{pmatrix} 2 & 3 \end{pmatrix}$$

The domain vector

$$\vec{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_2}(\vec{v}) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

has this image.

$$\mathrm{Rep}_{\mathcal{E}_1}(h(\vec{v})) = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \end{pmatrix}_{\mathcal{E}_1}$$

Since this is a representation with respect to the standard basis $\mathcal{E}_1$, meaning that vectors represent themselves, the image is $h(\vec{v}) = 2$.

Any Matrix Represents a Linear Map

The prior subsection shows how to start with a linear map and produce its matrix representation. What about the converse?

Example Fix a matrix

$$H = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

and also fix a domain and codomain, with bases.

$$\mathcal{E}_2 \subset \mathbb{R}^2 \quad \langle 1 - x, 1 + x \rangle \subset \mathcal{P}_1$$

Is there a linear map between the spaces associated with the matrix?

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Is there a linear map between the spaces associated with the matrix?

Consider $h\colon \mathbb{R}^2 \to \mathcal{P}_1$ defined by: for any domain vector $\vec{v}$, represent it with respect to the domain basis

$$\vec{v} = c_1\vec{e}_1 + c_2\vec{e}_2 \qquad \mathrm{Rep}_{E_2}(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

multiply that representation by H

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ 3c_1 + 4c_2 \end{pmatrix}$$

and then call $h(\vec{v})$ the codomain vector represented by the result.

$$h(\vec{v}) = (c_1 + 2c_2) \cdot (1 - x) + (3c_1 + 4c_2) \cdot (1 + x)$$

Observe that h is a function, that is, it is well-defined — for a given input $\vec{v}$, the output $h(\vec{v})$ exists and is unique. This is because the representation of a vector with respect to a basis is unique.

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We will now verify that h is a linear function. Fix domain vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$ and represent them with respect to the domain basis. Multiply $c \cdot \mathrm{Rep}_B(\vec{u}) + d \cdot \mathrm{Rep}_D(\vec{w})$ by H.

$$
\begin{aligned}
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} (c \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + d \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} cu_1 + dv_1 \\ cu_2 + dv_2 \end{pmatrix} \\
&= \begin{pmatrix} 1(cu_1 + dv_1) + 2(cu_2 + dv_2) \\ 3(cu_1 + dv_1) + 4(cu_2 + dv_2) \end{pmatrix} \\
&= \begin{pmatrix} 1cu_1 + 2cu_2 \\ 3cu_1 + 4cu_2 \end{pmatrix} + \begin{pmatrix} 1dv_1 + 2dv_2 \\ 3dv_1 + 4dv_2 \end{pmatrix} \\
&= c \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + d \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}
\end{aligned}
$$

By the definition of h, the result is $c \cdot \mathrm{Rep}_D(h(\vec{u})) + d \cdot \mathrm{Rep}_D(h(\vec{v}))$.

2.2 *Theorem* Any matrix represents a homomorphism between vector spaces of appropriate dimensions, with respect to any pair of bases.

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Proof We must check that for any matrix H and any domain and codomain bases B, D, the defined map h is linear. If $\vec{v}, \vec{u} \in V$ are such that

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \mathrm{Rep}_B(\vec{u}) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

and $c, d \in \mathbb{R}$ then the calculation

$$\begin{aligned} h(c\vec{v} + d\vec{u}) &= \big(h_{1,1}(cv_1 + du_1) + \cdots + h_{1,n}(cv_n + du_n)\big) \cdot \vec{\delta}_1 + \\ &\qquad \cdots + \big(h_{m,1}(cv_1 + du_1) + \cdots + h_{m,n}(cv_n + du_n)\big) \cdot \vec{\delta}_m \\ &= c \cdot h(\vec{v}) + d \cdot h(\vec{u}) \end{aligned}$$

supplies that check. QED

Finding Map Facts From the Matrix

We will show how to study the map by using an associated matrix. That is, to understand the map our algorithm is to fix bases and work with the resulting representations. Much of what we cover will be review, but putting it all in one place is useful.

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This equation is the key.

$$\mathrm{Rep}_{B,D}(h) \cdot \mathrm{Rep}_{B}(\vec{v}) = \mathrm{Rep}_{D}(h(\vec{v})) \qquad (*)$$

In this section we will write the domain vector $\vec{v}$ using the entries x, y, etc., and write the codomain vector $h(\vec{v})$ with a, b, etc.

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Example First, the number of rows and columns. This is an instance of $(*)$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

For matrix-vector multiplication to be defined, this 2×3 matrix must multiply a 3-tall input and get a 2-tall output. So, given a matrix H, the dimension of h's domain is the number of columns in H, and the dimension of h's codomain is the number of rows.

Next, an easy one: observe that $h\colon V \to W$ is the zero map $\vec{v} \mapsto \vec{0}$ if and only if it is represented, with respect to any bases, by the zero matrix.

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For one direction, assume that h is the zero map. Then for any bases B, D we have $h(\vec{\beta}_i) = \vec{0}_W$, which is represented with respect to D by the column vector of zeroes. For the other direction assume that there are bases B, D such that $\operatorname{Rep}_{B,D}(h)$ is the zero matrix. For each $\vec{\beta}_i$ we have that $\operatorname{Rep}_D(h(\vec{\beta}_1))$ is a vector of zeros, and so $h(\vec{\beta}_i)$ is $\vec{0}_W$. Extend linearly.

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Example The zero map $z\colon \mathbb{R}^2 \to \mathbb{R}^3$ is represented, with respect to any pair of bases, by the 2×3 zero matrix.

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Rank

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Proof Suppose that the matrix H is $m \times n$. Fix domain and codomain spaces V and W of dimension n and m with bases $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and D. Then H represents some linear map h between those spaces with respect to these bases whose range space

$$\begin{aligned}
\{ h(\vec{v}) \mid \vec{v} \in V \} &= \{ h(c_1 \vec{\beta}_1 + \cdots + c_n \vec{\beta}_n) \mid c_1, \ldots, c_n \in \mathbb{R} \} \\
&= \{ c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n) \mid c_1, \ldots, c_n \in \mathbb{R} \}
\end{aligned}$$

is the span $[\{ h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n) \}]$. The rank of the map h is the dimension of this range space.

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$$\begin{aligned}\{h(\vec{v}) \mid \vec{v} \in V\} &= \{h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) \mid c_1, \dots, c_n \in \mathbb{R}\} \\ &= \{c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n) \mid c_1, \dots, c_n \in \mathbb{R}\}\end{aligned}$$

is the span $[\{h(\vec{\beta}_1), \dots, h(\vec{\beta}_n)\}]$. The rank of the map h is the dimension of this range space.

The rank of the matrix is the dimension of its column space, the span of the set of its columns $[\{\mathrm{Rep}_D(h(\vec{\beta}_1)), \dots, \mathrm{Rep}_D(h(\vec{\beta}_n))\}]$.

To see that the two spans have the same dimension, recall from the proof of Lemma I.2.5 that if we fix a basis then representation with respect to that basis gives an isomorphism $\mathrm{Rep}_D\colon W \to \mathbb{R}^m$. Under this isomorphism there is a linear relationship among members of the range space if and only if the same relationship holds in the column space, e.g, $\vec{0} = c_1 \cdot h(\vec{\beta}_1) + \cdots + c_n \cdot h(\vec{\beta}_n)$ if and only if $\vec{0} = c_1 \cdot \mathrm{Rep}_D(h(\vec{\beta}_1)) + \cdots + c_n \cdot \mathrm{Rep}_D(h(\vec{\beta}_n))$. Hence, a subset of the range space is linearly independent if and only if the corresponding subset of the column space is linearly independent. Therefore the size of the largest linearly independent subset of the range space equals the size of the largest linearly independent subset of the column space, and so the two spaces have the same dimension. QED

Example Consider the linear transformation $t \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 2a - b \\ 2a - b \end{pmatrix}$$

Its range is the line $x = y$ and so the rank of the map is 1.

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We will see two matrices representing this map, the first with respect to these.

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rangle, \quad D = \langle \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} \rangle$$

This calculation is routine.

$$\mathrm{Rep}_D(t(\vec{\beta}_1)) = \mathrm{Rep}_D(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \mathrm{Rep}_D(\begin{pmatrix} -3 \\ -3 \end{pmatrix}) = \begin{pmatrix} -6 \\ -9 \end{pmatrix}$$

The representing matrix has matrix rank 1 since the second row is $3/2$ of the first.

$$\mathrm{Rep}_{B,D}(t) = \begin{pmatrix} 2 & -6 \\ 3 & -9 \end{pmatrix}$$

Still considering $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 2a-b \\ 2a-b \end{pmatrix}$$

this time represent it with respect to the standard bases $\mathcal{E}_3, \mathcal{E}_3$.

$$t(\vec{e}_1) = t(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_2}(\begin{pmatrix} 2 \\ 2 \end{pmatrix}) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$
$$t(\vec{e}_2) = t(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_2}(\begin{pmatrix} -1 \\ -1 \end{pmatrix}) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

(Remember that with respect to the standard basis, a vector represents itself.)

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t) = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$$

Gauss's Method on that matrix ends with one nonzero row so it has matrix rank 1, as on the prior slide.

One-to-One and Onto

2.6 *Corollary* Let h be a linear map represented by a matrix H. Then h is onto if and only if the rank of H equals the number of its rows, and h is one-to-one if and only if the rank of H equals the number of its columns.

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Proof For the onto half, the dimension of the range space of h is the rank of h, which equals the rank of H by the theorem. Since the dimension of the codomain of h equals the number of rows in H, if the rank of H equals the number of rows then the dimension of the range space equals the dimension of the codomain. But a subspace with the same dimension as its superspace must equal that superspace (because any basis for the range space is a linearly independent subset of the codomain whose size is equal to the dimension of the codomain, and thus this basis for the range space must also be a basis for the codomain).

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For the other half, a linear map is one-to-one if and only if it is an isomorphism between its domain and its range, that is, if and only if its domain has the same dimension as its range. The number of columns in H is the dimension of h's domain and by the theorem the rank of H equals the dimension of h's range. QED

Example Any transformation rotating vectors counterclockwise by Θ radians $t_\Theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ is represented with respect to the standard bases by this matrix.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t_\Theta) = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix}$$

The $\Theta = \pi/4$ instance is

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t_{\pi/4}) = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} = (\sqrt{2}/2) \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and the rank of this matrix is two (we can see by inspection that Gauss's Method will result in two non-zero rows). This reflects the geometry, which shows that the map $t_{\pi/4}$ is one-to-one and onto.

2.7 *Definition* A linear map that is one-to-one and onto is *nonsingular*, otherwise it is *singular*. That is, a linear map is nonsingular if and only if it is an isomorphism.

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Proof Assume that the map $h\colon V \to W$ is nonsingular. Corollary 2.6 says that for any matrix H representing that map, because h is onto the number of rows of H equals the rank of H, and because h is one-to-one the number of columns of H is also equal to the rank of H. Hence H is square.

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Next assume that H is square, $n \times n$. The matrix H is nonsingular if and only if its row rank is n, which is true if and only if H's rank is n by Theorem Two.III.3.11 , which is true if and only if h's rank is n by Theorem 2.4 , which is true if and only if h is an isomorphism by Theorem I.2.3 . (This last holds because the domain of h is n-dimensional as it is the number of columns in H.) QED

Example This matrix

$$\begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}$$

is nonsingular since its two rows form a linearly independent set (we see that by inspection). So any map, with any domain and codomain, and represented by this matrix with respect to any pair of bases, is an isomorphism.

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Example Gauss's method shows that this matrix

$$\begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 1 \\ -1 & 0 & 5 \end{pmatrix}$$

is singular so any map that it represents will be a homomorphism that is not an isomorphism.

Computing Range and Null Spaces

Example Consider this linear map $h\colon \mathbb{R}^3 \to \mathbb{R}^3$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ x+z \\ x-z \end{pmatrix}$$

A function is onto if every codomain member $\vec{w}$ is the image of at least one domain member $\vec{v}$. This function is not onto because there is a restriction: the first component of the output must be the average of the other two.

A function is one-to-one if every codomain member $\vec{w}$ is the image of at most one domain member $\vec{v}$. This function is not one-to-one because the output entries don't use y, so holding x and z constant and varying y gives many inputs that are all associated with the same output.

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We could decide onto-ness and one-to-one-ness for this function but maybe we will one day meet a function we can't beat. We want an algorithm.

For the calculations we use $\mathcal{E}_3 \subseteq \mathbb{R}^3$ for both the domain and codomain. Find the action of h on the domain's basis

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\1\\1\end{pmatrix} \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

and represent those outputs with respect to the codomain's basis.

$$\mathrm{Rep}_{\mathcal{E}_3}(\begin{pmatrix}1\\1\\1\end{pmatrix}) = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad \mathrm{Rep}_{\mathcal{E}_3}(\begin{pmatrix}0\\0\\0\end{pmatrix}) = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad \mathrm{Rep}_{\mathcal{E}_3}(\begin{pmatrix}0\\1\\-1\end{pmatrix}) = \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

(The standard basis makes for easy calculations.)

$$H = \mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(h) = \begin{pmatrix}1 & 0 & 0\\1 & 0 & 1\\1 & 0 & -1\end{pmatrix}$$

This is the instance of $\mathrm{Rep}_{B,D}(h) \cdot \mathrm{Rep}_B(\vec{v})$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

To find the range space solve for x, y, and z.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 1 & 0 & 1 & b \\ 1 & 0 & -1 & c \end{array}\right) \xrightarrow[-\rho_1+\rho_3]{-\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 0 & 1 & -a+b \\ 0 & 0 & -1 & -a+c \end{array}\right)$$

$$\xrightarrow{\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 0 & 1 & -a+b \\ 0 & 0 & 0 & -2a+b+c \end{array}\right)$$

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$$\xrightarrow{\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 0 & 1 & -a+b \\ 0 & 0 & 0 & -2a+b+c \end{array}\right)$$

The rank is 2. So h is not onto: specifically, the range space $\mathscr{R}(h)$ consists of only those $\vec{w}$'s whose representations satisfy that $0 = -2a + b + c$.

To decide if h is one-to-one we solve the associated homogeneous system.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \longrightarrow \quad \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set

$$\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = 0 \text{ and } z = 0 \} = \{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot y \mid y \in \mathbb{R} \}$$

is nontrivial so h is not one-to-one. Specifically, the null space $\mathscr{N}(h)$ is the set of domain vectors whose representation falls in that one-dimensional set.

Example Let h be the transformation of $\mathbb{R}^2$ represented with respect to the standard basis by this matrix.

$$H = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

This matrix has rank 2 since the second row is not a multiple of the first.

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$$H = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

This matrix has rank 2 since the second row is not a multiple of the first. This is the instance of equation $(*)$.

$$\begin{gathered} \mathrm{Rep}_{B,D}(h) \cdot \mathrm{Rep}_B(\vec{v}) = \mathrm{Rep}_D(\vec{w}) \\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \end{gathered}$$

Do the calculation to solve this linear system for x and y.

$$\left(\begin{array}{cc|c} 1 & 2 & a \\ 3 & 4 & b \end{array}\right) \xrightarrow{-3\rho_1+\rho_2} \left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & -2 & -3a+b \end{array}\right)$$
$$\xrightarrow{-(1/2)\rho_2} \left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & (3/2)a-(1/2)b \end{array}\right) \xrightarrow{-2\rho_2+\rho_1} \left(\begin{array}{cc|c} 1 & 0 & -2a+b \\ 0 & 1 & (3/2)a-(1/2)b \end{array}\right)$$

(Continued from prior slide.)

$$\left(\begin{array}{cc|c} 1 & 2 & a \\ 3 & 4 & b \end{array}\right) \xrightarrow{-3\rho_1+\rho_2} \quad \cdots \quad \left(\begin{array}{cc|c} 1 & 0 & -2a+b \\ 0 & 1 & (3/2)a-(1/2)b \end{array}\right)$$

The calculation shows that for all codomain vectors

$$\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}$$

there is an associated domain $\vec{v}$, namely with $x = -2a + b$ and $y = (3/2)a - (1/2)b$. So h is onto, $\mathscr{R}(h) = \mathbb{R}^2$, and the rank of h is as large as it could be, 2.

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Doing the homogeneous system by substituting $a = 0$, $b = 0$ above shows the null space is trivial. So h is one-to-one;

$$\mathscr{N}(h) = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$$

The trivial null space has an empty basis, so the nullity is 0.

Example Suppose that a map h between real spaces is represented by this.

$$H = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \end{pmatrix}$$

The domain has dimension 3 and the codomain has dimension 2 so $h\colon \mathbb{R}^3 \to \mathbb{R}^2$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 2 & 3 & 0 & b \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \xrightarrow{-\rho_2} \xrightarrow{-2\rho_2+\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & -3 & -3a+2b \\ 0 & 1 & 2 & 2a-b \end{array}\right)$$

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For each a, b there is at least one triple x, y, z so this map is onto; $\text{rank}(h) = 2$.

For each a, b there is more than one triple x, y, z so this map is not one-to-one. Said another way, find the null space by substituting $a = 0$, $b = 0$ above to solve the homogeneous system.

$$\mathscr{N}(h) = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid y + 2z = 0 \text{ and } x - 3z = 0 \} = \{ \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} z \mid z \in \mathbb{R} \}$$

The nullity is 1.

Leverage

So we can determine things about a map by fixing bases and calculating with the representation.

We close with an example showing how to extend those calculations to the underlying spaces.

Example Fix these, as well as the domain and codomain $\mathbb{R}^2$ and $\mathcal{P}_2$.

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle, D = \langle 1+x, 1-x, x+x^2 \rangle \quad H = \mathrm{Rep}_{B,D}(h) = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$$

Gauss's Method is routine.

$$\left(\begin{array}{cc|c} 1 & 2 & a \\ 1 & 3 & b \\ 1 & 4 & c \end{array}\right) \xrightarrow[-\rho_1+\rho_3]{-\rho_1+\rho_2} \xrightarrow{-2\rho_2+\rho_3} \xrightarrow{-2\rho_2+\rho_1} \left(\begin{array}{cc|c} 1 & 0 & 3a-2b \\ 0 & 1 & -a+b \\ 0 & 0 & a-2b+c \end{array}\right)$$

The null space is trivial $\mathscr{N}(h) = \{\vec{0}\}$ so h is one-to-one. The range contains the output vectors whose representations satisfy $a - 2b + c = 0$, so h is not onto.

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The null space is trivial $\mathscr{N}(h) = \{\vec{0}\}$ so h is one-to-one. The range contains the output vectors whose representations satisfy $a-2b+c=0$, so h is not onto. To get the underlying elements of the codomain $\mathcal{P}_2$, expand.

$$\begin{aligned}
\mathscr{R}(h) &= \{\vec{p} \in \mathcal{P}_2 \mid \mathrm{Rep}_D(\vec{p}) = \begin{pmatrix} 2b-c \\ b \\ c \end{pmatrix}\} \\
&= \{(2b-c)\cdot(1+x) + b\cdot(1-x) + c\cdot(x+x^2) \mid b, c \in \mathbb{R}\} \\
&= \{b\cdot(3+x) + c\cdot(-1+x^2) \mid b, c \in \mathbb{R}\}
\end{aligned}$$

That's the span of $\langle 3+x, -1+x^2 \rangle$, which is linearly independent so it is a basis.