Four.I Definition of Determinant

Linear Algebra

Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Properties of Determinants

Nonsingular matrices

For any matrix, whether or not it is nonsingular is a key question. Recall that an $n \times n$ matrix T is nonsingular if and only if each of these holds:

- any system $T\vec{x} = \vec{b}$ has a solution and that solution is unique;
- Gauss-Jordan reduction of T yields an identity matrix;
- the rows of T form a linearly independent set;
- the columns of T form a linearly independent set, a basis for $\mathbb{R}^n$;
- any map that T represents is an isomorphism;
- an inverse matrix T^{-1} exists.

This chapter develops a formula to determine whether a matrix is nonsingular.

Determining nonsingularity is trivial for 1×1 matrices.

$$\begin{pmatrix} a \end{pmatrix} \quad \text{is nonsingular iff} \quad a \neq 0$$

Corollary Three.IV.4.11 gives the 2×2 formula.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{is nonsingular iff} \quad ad - bc \neq 0$$

We can produce the 3×3 formula as we did the prior one, although the computation is intricate (see Exercise 10).

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{is nonsingular iff} \quad aei + bfg + cdh - hfa - idb - gec \neq 0$$

With these cases in mind, we posit a family of formulas: a, $ad - bc$, etc. For each n the formula defines a *determinant* function $\det_{n \times n} : \mathcal{M}_{n \times n} \to \mathbb{R}$ such that an $n \times n$ matrix T is nonsingular if and only if $\det_{n \times n}(T) \neq 0$.

The plan

The prior slide gives a formula for the 1×1, 2×2, and 3×3 determinants. But while those three formulas are a help, they don't make clear what should be the general formula, for the $n\times n$ case.

So we will proceed by stating some conditions that a determinant function must satisfy. The conditions extrapolate from the 1×1, 2×2, and 3×3 cases; see the book's discussion. They will let us compute the determinant of a square matrix via Gauss's Method, which we know to be fast and easy.

However, defining the determinant function by giving a list of conditions has a downside. We must verify this function is well-defined, that there is at least one function satisfying those conditions and also that there is only one such. To verify that, after giving the algorithm we will develop a formula, the permutation expansion, satisfying the conditions.

Definition of determinant

2.1 *Definition* A $n \times n$ *determinant* is a function $\det\colon \mathcal{M}_{n\times n} \to \mathbb{R}$ such that

1) $\det(\vec{\rho}_1, \ldots, k\cdot\vec{\rho}_i + \vec{\rho}_j, \ldots, \vec{\rho}_n) = \det(\vec{\rho}_1, \ldots, \vec{\rho}_j, \ldots, \vec{\rho}_n)$ for $i \neq j$
2) $\det(\vec{\rho}_1, \ldots, \vec{\rho}_j, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_n) = -\det(\vec{\rho}_1, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_j, \ldots, \vec{\rho}_n)$ for $i \neq j$
3) $\det(\vec{\rho}_1, \ldots, k\vec{\rho}_i, \ldots, \vec{\rho}_n) = k\cdot\det(\vec{\rho}_1, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_n)$ for any scalar k
4) $\det(I) = 1$ where I is an identity matrix

(the $\vec{\rho}$'s are the rows of the matrix). We often write $|T|$ for $\det(T)$.

2.2 *Remark* Condition (2) is redundant since

$$T \xrightarrow{\rho_i+\rho_j} \xrightarrow{-\rho_j+\rho_i} \xrightarrow{\rho_i+\rho_j} \xrightarrow{-\rho_i} \hat{T}$$

swaps rows i and j. We have listed it for consistency with the Gauss's Method presentation in earlier chapters.

Consequences of the definition

2.4 *Lemma* A matrix with two identical rows has a determinant of zero. A matrix with a zero row has a determinant of zero. A matrix is nonsingular if and only if its determinant is nonzero. The determinant of an echelon form matrix is the product down its diagonal.

Proof To verify the first sentence swap the two equal rows. The sign of the determinant changes but the matrix is the same and so its determinant is the same. Thus the determinant is zero.

For the second sentence multiply the zero row by two. That doubles the determinant but it also leaves the row unchanged, and hence leaves the determinant unchanged. Thus the determinant must be zero.

Do Gauss-Jordan reduction for the third sentence, $T \rightarrow \cdots \rightarrow \hat{T}$. By the first three properties the determinant of T is zero if and only if the determinant of $\hat{T}$ is zero (although the two could differ in sign or magnitude). A nonsingular matrix T Gauss-Jordan reduces to an identity matrix and so has a nonzero determinant. A singular T reduces to a $\hat{T}$ with a zero row; by the second sentence of this lemma its determinant is zero.

The fourth sentence has two cases. If the echelon form matrix is singular then it has a zero row. Thus it has a zero on its diagonal and the product down its diagonal is zero. By the third sentence of this result the determinant is zero and therefore this matrix's determinant equals the product down its diagonal.

If the echelon form matrix is nonsingular then none of its diagonal entries is zero. This means that we can divide by those entries and use condition (3) to get 1's on the diagonal.

$$\begin{vmatrix} t_{1,1} & t_{1,2} & & t_{1,n} \\ 0 & t_{2,2} & & t_{2,n} \\ & & \ddots & \\ 0 & & & t_{n,n} \end{vmatrix} = t_{1,1} \cdot t_{2,2} \cdots t_{n,n} \cdot \begin{vmatrix} 1 & t_{1,2}/t_{1,1} & & t_{1,n}/t_{1,1} \\ 0 & 1 & & t_{2,n}/t_{2,2} \\ & & \ddots & \\ 0 & & & 1 \end{vmatrix}$$

Then the Jordan half of Gauss-Jordan elimination leaves the identity matrix.

$$= t_{1,1} \cdot t_{2,2} \cdots t_{n,n} \cdot \begin{vmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{vmatrix} = t_{1,1} \cdot t_{2,2} \cdots t_{n,n} \cdot 1$$

So in this case also, the determinant is the product down the diagonal.

QED

The conditions allow us to compute the determinant of a matrix using Gauss's Method

Example On this matrix we can perform the Gauss's Method steps of $-2\rho_1+\rho_2$ and $-3\rho_1+\rho_3$, followed by $-(5/3)\rho_2+\rho_3$. Condition (1) says that these row combination operations leave the determinant unchanged.

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & 0 & 4 \\ 3 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 \\ 0 & -6 & -8 \\ 0 & -10 & -11 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 \\ 0 & -6 & -8 \\ 0 & 0 & -7/3 \end{vmatrix}$$

The determinant is the product down the diagonal: $1\cdot(-6)\cdot(-7/3)=14$.

Example To see what happens with a row swap, consider this matrix. The swap changes the determinant's sign.

$$\begin{vmatrix} 0 & 3 & 1 \\ 1 & 2 & 0 \\ 1 & 5 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 5 & 2 \end{vmatrix}$$

Finish by performing $-\rho_1 + \rho_3$ followed by $-\rho_2 + \rho_3$

$$= -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

and then multiplying down the diagonal. The determinant of the original matrix is -3.

Example Finally, to illustrate condition (3) contrast these two.

$$\begin{vmatrix} 5 & 10 \\ 3 & 4 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

Condition (3) gives

$$\begin{vmatrix} 5 & 10 \\ 3 & 4 \end{vmatrix} = 5 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

and by performing $-3\rho_1 + \rho_2$ and multiplying down the diagonal we get this.

$$= 5 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 5 \cdot \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = 5 \cdot (-2) = -10$$

Thus here is the contrast.

$$\begin{vmatrix} 5 & 10 \\ 3 & 4 \end{vmatrix} = -10 \qquad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

Warning

The formula for the determinant of a 2×2 matrix has something to do with multiplying along the diagonals.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Sometimes people have learned a mnemonic for the 3×3 formula that also has to do with multplying diagonals.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = {\color{green} aei + bfg + cdh} {\color{red} - gec - hfa - idb}$$

$$\left(\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right)$$

Don't try to extend to 4×4 or larger sizes. Instead, for larger matrices use Gauss's Method.

Existence and Uniqueness

As described before the definition of determinant, defining the function by giving conditions that it must satisfy has a downside. We must verify that there is a function satisfying those conditions, and that there is only one such function.

As to existence, how could there fail to be such a function, when we have been computing its outputs? Consider these two Gauss's Method reductions of the same matrix, the first without any row swap

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{-3\rho_1+\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$

and the second with one.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{\rho_1\leftrightarrow\rho_2} \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \xrightarrow{-(1/3)\rho_1+\rho_2} \begin{pmatrix} 3 & 4 \\ 0 & 2/3 \end{pmatrix}$$

Both yield the determinant -2 since in the second one we note that the row swap changes the sign of the result we get by multiplying down the diagonal. But just because we get consistent results in this case does not mean that all determinant computations give the same value.

Below we will give an alternative way to compute the determinant, a formula. It will make plain that the determinant is a function, that it returns well-defined outputs. Computing a determinant with this formula is not practical since it is very slow but it is nonetheless invaluable for the theory.

But before that we will show uniqueness, that if a function satisfying the conditions exists then there is only one.

2.7 *Lemma* For each n, if there is an $n \times n$ determinant function then it is unique.

Proof Suppose that there are two functions $\det_1, \det_2 \colon \mathcal{M}_{n\times n} \to \mathbb{R}$ satisfying the properties of Definition 2.1 and its consequence Lemma 2.4 . Given a square matrix M, fix some way of performing Gauss's Method to bring the matrix to echelon form (it does not matter that there are multiple ways, just fix one of them). By using this fixed reduction as in the above examples — keeping track of row-scaling factors and how the sign alternates on row swaps, and then multiplying down the diagonal of the echelon form result — we can compute the value that these two functions must return on M, and they must return the same value. Since they give the same output on every input, they are the same function. QED

The Permutation Expansion

The determinant function is not linear

Example The determinant does not in general satisfy that $\det(k \cdot T) = k \cdot \det(T)$. The second matrix here is twice the first but the determinant does not double.

$$\begin{vmatrix} 3 & -3 & 9 \\ 1 & -1 & 7 \\ 2 & 4 & 0 \end{vmatrix} = -72 \qquad \begin{vmatrix} 6 & -6 & 18 \\ 2 & -2 & 14 \\ 4 & 8 & 0 \end{vmatrix} = -576$$

Condition (3) has the determinant scale one row at a time.

$$\begin{aligned} \begin{vmatrix} 6 & -6 & 18 \\ 2 & -2 & 14 \\ 4 & 8 & 0 \end{vmatrix} &= 2 \cdot \begin{vmatrix} 3 & -3 & 9 \\ 2 & -2 & 14 \\ 4 & 8 & 0 \end{vmatrix} \\ &= 4 \cdot \begin{vmatrix} 3 & -3 & 9 \\ 1 & -1 & 7 \\ 4 & 8 & 0 \end{vmatrix} \\ &= 8 \cdot \begin{vmatrix} 3 & -3 & 9 \\ 1 & -1 & 7 \\ 2 & 4 & 0 \end{vmatrix} \end{aligned}$$

So determinants are not linear: with scalar multiplication, the scalars come out one row at a time. What happens with addition?

Consider condition (3) applied to the case $k = 2$.

$$\det(\vec{\rho}_1, \ldots, 2\vec{\rho}_i, \ldots, \vec{\rho}_n) = 2 \cdot \det(\vec{\rho}_1, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_n)$$

Rewrite it as addition.

$$\begin{aligned}\det(\vec{\rho}_1, \ldots, \vec{\rho}_i + \vec{\rho}_i, \ldots, \vec{\rho}_n)& \\ &= \det(\vec{\rho}_1, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_n) + \det(\vec{\rho}_1, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_n)\end{aligned}$$

Of course this extends to $k = 3$, etc.

This hints that besides bringing out scalars one row at a time, determinants also break along a plus sign one row at a time.

Multilinear

3.2 *Definition* Let V be a vector space. A map $f\colon V^n \to \mathbb{R}$ is *multilinear* if

1) $f(\vec{\rho}_1,\ldots,\vec{v}+\vec{w},\ldots,\vec{\rho}_n) = f(\vec{\rho}_1,\ldots,\vec{v},\ldots,\vec{\rho}_n) + f(\vec{\rho}_1,\ldots,\vec{w},\ldots,\vec{\rho}_n)$
2) $f(\vec{\rho}_1,\ldots,k\vec{v},\ldots,\vec{\rho}_n) = k\cdot f(\vec{\rho}_1,\ldots,\vec{v},\ldots,\vec{\rho}_n)$

for $\vec{v},\vec{w}\in V$ and $k\in\mathbb{R}$.

3.3 *Lemma* Determinants are multilinear.

Proof Property (2) here is just Definition 2.1 's condition (3) so we need only verify property (1).

There are two cases. If the set of other rows $\{\vec{\rho}_1,\ldots,\vec{\rho}_{i-1},\vec{\rho}_{i+1},\ldots,\vec{\rho}_n\}$ is linearly dependent then all three matrices are singular and so all three determinants are zero and the equality is trivial.

Therefore assume that the set of other rows is linearly independent. We can make a basis by adding one more vector $\langle \vec{\rho}_1, \ldots, \vec{\rho}_{i-1}, \vec{\beta}, \vec{\rho}_{i+1}, \ldots, \vec{\rho}_n \rangle$. Express $\vec{v}$ and $\vec{w}$ with respect to this basis

$$\begin{aligned}
\vec{v} &= v_1 \vec{\rho}_1 + \cdots + v_{i-1} \vec{\rho}_{i-1} + v_i \vec{\beta} + v_{i+1} \vec{\rho}_{i+1} + \cdots + v_n \vec{\rho}_n \\
\vec{w} &= w_1 \vec{\rho}_1 + \cdots + w_{i-1} \vec{\rho}_{i-1} + w_i \vec{\beta} + w_{i+1} \vec{\rho}_{i+1} + \cdots + w_n \vec{\rho}_n
\end{aligned}$$

and add.

$$\vec{v} + \vec{w} = (v_1 + w_1) \vec{\rho}_1 + \cdots + (v_i + w_i) \vec{\beta} + \cdots + (v_n + w_n) \vec{\rho}_n$$

Consider the left side of (1) and expand $\vec{v} + \vec{w}$.

$$\det(\vec{\rho}_1, \ldots, (v_1 + w_1) \vec{\rho}_1 + \cdots + (v_i + w_i) \vec{\beta} + \cdots + (v_n + w_n) \vec{\rho}_n, \ldots, \vec{\rho}_n) \quad (*)$$

By the definition of determinant's condition (1), the value of $(*)$ is unchanged by the operation of adding $-(v_1 + w_1) \vec{\rho}_1$ to the i-th row $\vec{v} + \vec{w}$. The i-th row becomes this.

$$\vec{v} + \vec{w} - (v_1 + w_1) \vec{\rho}_1 = (v_2 + w_2) \vec{\rho}_2 + \cdots + (v_i + w_i) \vec{\beta} + \cdots + (v_n + w_n) \vec{\rho}_n$$

Next add $-(v_2 + w_2)\vec{\rho}_2$, etc., to eliminate all of the terms from the other rows. Apply condition (3) from the definition of determinant.

$$\begin{aligned}
&\det(\vec{\rho}_1, \ldots, \vec{v} + \vec{w}, \ldots, \vec{\rho}_n) \\
&\qquad = \det(\vec{\rho}_1, \ldots, (v_i + w_i) \cdot \vec{\beta}, \ldots, \vec{\rho}_n) \\
&\qquad = (v_i + w_i) \cdot \det(\vec{\rho}_1, \ldots, \vec{\beta}, \ldots, \vec{\rho}_n) \\
&\qquad = v_i \cdot \det(\vec{\rho}_1, \ldots, \vec{\beta}, \ldots, \vec{\rho}_n) + w_i \cdot \det(\vec{\rho}_1, \ldots, \vec{\beta}, \ldots, \vec{\rho}_n)
\end{aligned}$$

Now this is a sum of two determinants. To finish, bring v_i and w_i back inside in front of the $\vec{\beta}$'s and use row combinations again, this time to reconstruct the expressions of $\vec{v}$ and $\vec{w}$ in terms of the basis. That is, start with the operations of adding $v_1\vec{\rho}_1$ to $v_i\vec{\beta}$ and $w_1\vec{\rho}_1$ to $w_i\vec{\rho}_1$, etc., to get the expansions of $\vec{v}$ and $\vec{w}$. QED

(*Remark.* Some authors use multilinearity to define the determinant in place of our four conditions that lead to Gauss's Method.)

Multilinearity breaks a determinant into a sum of simple ones.

Example We can expand this determinant

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

along the first row

$$= \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 3 & 4 \end{vmatrix}$$

and then expand both of those on second row.

$$= \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 0 & 4 \end{vmatrix}$$

Each matrix is simple in that of its rows are all zeroes except for a single entry from the starting matrix.

Of these four, the first and last are 0 because the matrices are nonsingular, since they have a second row that is a multiple of the first. We are left with two determinants, where in each the matrix is all zeros except for one entry from the starting matrix in each row and each column. We'll show our strategy for evaluating these determinants below.

Example Similarly we can use multilinearity to expand this determinant.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 7 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 9 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 7 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 8 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 6 \\ 7 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 8 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & 2 & 0 \\ 4 & 0 & 0 \\ 7 & 0 & 0 \end{vmatrix} + \qquad \cdots \qquad + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

In each of the $3^3 = 27$ determinants on the right the matrix is all zeros but for a single entry from the starting matrix in each row.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 7 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{vmatrix} + \cdots + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 0 & 6 \\ 0 & 8 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

For any of these, if two of the matrix rows have their original matrix entries in the same column then the determinant is 0 since then one matrix row is a multiple of the other.

We've reduced to a sum of determinants, where each matrix is all 0's but for a single entry from the original in each row and column. There are $3 \cdot 2 \cdot 1 = 6$ of these.

$$
\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 8 & 0 \end{vmatrix} \\
&\quad + \begin{vmatrix} 0 & 2 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 9 \end{vmatrix} + \begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 6 \\ 7 & 0 & 0 \end{vmatrix} \\
&\quad + \begin{vmatrix} 0 & 0 & 3 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 5 & 0 \\ 7 & 0 & 0 \end{vmatrix} \\
&= 45 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 48 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\
&\quad + 72 \cdot \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 84 \cdot \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\
&\quad + 96 \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + 105 \cdot \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}
\end{aligned}
$$

After bringing out each entry from the original matrix, we are left with matrices that are all 0's except for a single 1 in each row and column.

Permutation matrices

Recall Definition Three.IV.3.14 , that a *permutation matrix* is square, with entries 0's except for a single 1 in each row and column. We now introduce a notation for permutation matrices.

3.7 *Definition* An *n-permutation* is a function on the first n positive integers $\phi\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ that is one-to-one and onto.

So, in a permutation each number 1, ..., n is the output associated with one and only one input. We sometimes denote a permutation as the sequence $\phi = \langle \phi(1), \phi(2), \ldots, \phi(n) \rangle$.

3.8 *Example* These are the 2-permutations.

$$\begin{array}{lll} \phi_1\colon & 1 \mapsto 1 & 2 \mapsto 2 \\ \phi_2\colon & 1 \mapsto 2 & 2 \mapsto 1 \end{array}$$

The sequence notation is shorter: $\phi_1 = \langle 1, 2 \rangle$ and $\phi_2 = \langle 2, 1 \rangle$.

3.9 *Example* In the sequence notation the 3-permutations are $\phi_1 = \langle 1, 2, 3 \rangle$, $\phi_2 = \langle 1, 3, 2 \rangle$, $\phi_3 = \langle 2, 1, 3 \rangle$, $\phi_4 = \langle 2, 3, 1 \rangle$, $\phi_5 = \langle 3, 1, 2 \rangle$, and $\phi_6 = \langle 3, 2, 1 \rangle$.

We denote the row vector that is all 0's except for a 1 in entry j with ι_j so that the four-wide ι_2 is $(0\ \ 1\ \ 0\ \ 0)$. Now our notation for permutation matrices is: with any $\phi = \langle \phi(1), \ldots, \phi(n) \rangle$ associate the matrix whose rows are $\iota_{\phi(1)}, \ldots, \iota_{\phi(n)}$.

Example Associated with the 4-permutation $\phi = \langle 2, 4, 3, 1 \rangle$ is the matrix whose rows are the matching ι's.

$$P_\phi = \begin{pmatrix} \iota_2 \\ \iota_4 \\ \iota_3 \\ \iota_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Permutation expansion

3.12 *Definition* The *permutation expansion* for determinants is

$$\begin{vmatrix} t_{1,1} & t_{1,2} & \ldots & t_{1,n} \\ t_{2,1} & t_{2,2} & \ldots & t_{2,n} \\ & \vdots & & \\ t_{n,1} & t_{n,2} & \ldots & t_{n,n} \end{vmatrix} = \begin{aligned} & t_{1,\phi_1(1)} t_{2,\phi_1(2)} \cdots t_{n,\phi_1(n)} |P_{\phi_1}| \\ & + t_{1,\phi_2(1)} t_{2,\phi_2(2)} \cdots t_{n,\phi_2(n)} |P_{\phi_2}| \\ & \vdots \\ & + t_{1,\phi_k(1)} t_{2,\phi_k(2)} \cdots t_{n,\phi_k(n)} |P_{\phi_k}| \end{aligned}$$

where $\phi_1, \ldots, \phi_k$ are all of the n-permutations.

We can restate the formula in *summation notation*

$$|T| = \sum_{\text{permutations } \phi} t_{1,\phi(1)} t_{2,\phi(2)} \cdots t_{n,\phi(n)} \, |P_\phi|$$

read aloud as, "the sum, over all permutations ϕ, of terms having the form $t_{1,\phi(1)} t_{2,\phi(2)} \cdots t_{n,\phi(n)} |P_\phi|$."

Example Recall that there are two 2-permutations $\phi_1 = \langle 1, 2\rangle$ and $\phi_2 = \langle 2, 1\rangle$. These are the associated permutation matrices

$$P_{\phi_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_{\phi_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

giving this expansion.

$$\begin{aligned} \begin{vmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{vmatrix} &= t_{1,1}t_{2,2} \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + t_{1,2}t_{2,1} \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ &= t_{1,1}t_{2,2} \cdot 1 + t_{1,2}t_{2,1} \cdot (-1) \end{aligned}$$

(From prior work we know that the determinant $|P_{\phi_2}|$ equals -1 because we can bring that to the identity matrix with one row swap.) Renaming the matrix entries gives the familiar 2×2 formula.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The only thing remaining in our process of finding a formula for the determinant (not involving Gauss's Method) is to give a formula for the determinant of such matrices. We do that in the next subsection.

The next subsection is optional so for those not going on, we state its results here. We will follow up on the second result.

3.14 *Theorem* For each n there is an $n \times n$ determinant function.

3.15 *Theorem* The determinant of a matrix equals the determinant of its transpose.

Column properties from row properties

The fact that the determinant of the transpose equals the determinant of the matrix allows us to transfer results: statements about rows become statements about columns.

For instance, the first condition in the definition of determinant is that a determinant is unchanged by a row combination: where A is square and $A \xrightarrow{k\rho_i+\rho_j} \hat{A}$ (with $i \neq j$) then $\det(A) = \det \hat{A}$. To see that the same holds for columns, observe that we can do a column combination by transposing, doing a row combination, and then transposing back. These three don't change the determinant.

3.16 *Corollary* A matrix with two equal columns is singular. Column swaps change the sign of a determinant. Determinants are multilinear in their columns.

Proof For the first statement, transposing the matrix results in a matrix with the same determinant, and with two equal rows, and hence a determinant of zero. Prove the other two in the same way. QED

Determinants Exist (optional)

Inversion

4.1 *Definition* In a permutation $\phi = \langle \ldots, k, \ldots, j, \ldots \rangle$, elements such that $k > j$ are in an *inversion* of their natural order. Similarly, in a permutation matrix two rows

$$P_\phi = \begin{pmatrix} \vdots \\ \iota_k \\ \vdots \\ \iota_j \\ \vdots \end{pmatrix}$$

such that $k > j$ are in an *inversion*.

Example The permutation $\phi = \langle 3, 2, 1 \rangle$ has three inversions: 3 is before 2, 3 is before 1, and 2 is before 1.

before	1	2
2	$\times$	
3	$\times$	$\times$

Example This matrix

$$P_\phi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \iota_2 \\ \iota_1 \\ \iota_4 \\ \iota_3 \end{pmatrix}$$

associated with the permutation $\phi = \langle 2, 1, 4, 3 \rangle$ has two inversions: row 2 comes before row 1, and row 4 is before row 3.

before	1	2	3
2	$\times$		
3			
4			$\times$

4.4 *Lemma* A row-swap in a permutation matrix changes the number of inversions from even to odd, or from odd to even.

The book contains the proof; we will illustrate. There are two cases, that the rows are adjacent and that they aren't.

Example Swapping adjacent rows

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{\rho_2 \leftrightarrow \rho_3} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

changes the number of inversions by one.

before	1	2	3
2			
3		$\times$	
4	$\times$	$\times$	$\times$

before	1	2	3
2			
3	$\times$	$\times$	
4	$\times$	$\times$	$\times$

Except for the ι_1 and ι_3 involved in the swap, all of the inter-row relationships are unchanged.

Example To swap non-adjacent rows

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_4} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

do a sequence of swaps of adjacent rows.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \xrightarrow{\rho_2 \leftrightarrow \rho_3} \xrightarrow{\rho_3 \leftrightarrow \rho_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\rho_3 \leftrightarrow \rho_2} \xrightarrow{\rho_2 \leftrightarrow \rho_1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That's an odd number of swaps, five of them, each of which flips the parity of the number of inversions. Altogether the parity changes, from even to odd.

Signum

4.7 *Definition* The *signum* of a permutation $\operatorname{sgn}(\phi)$ is -1 if the number of inversions in ϕ is odd and is $+1$ if the number of inversions is even.

Example The permutation $\phi = \langle 3, 2, 1 \rangle$ associated with this matrix

$$P_\phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

has an odd number of inversions, three.

before	1	2
2	$\times$	
3	$\times$	$\times$

The signum is $\operatorname{sgn}(\phi) = -1$.

Example The permutation $\phi = \langle 3, 2, 4, 1 \rangle$ has four inversions: 3 is before 2 and 1, 2 is before 1, and 4 is before 1. So $\operatorname{sgn}(\phi) = +1$.

4.5 *Corollary* If a permutation matrix has an odd number of inversions then swapping it to the identity takes an odd number of swaps. If it has an even number of inversions then swapping to the identity takes an even number.

Proof The identity matrix has zero inversions. To change an odd number to zero requires an odd number of swaps, and to change an even number to zero requires an even number of swaps. QED

Plan accomplished

We are in the process of showing that a function exists that satisfies the four conditions in the definition of determinant. We must show that for each input square matrix there is a well-defined output value — Gauss's Method can be done in more than one way so it isn't obvious that by keeping track of signs and multiplying down the diagonal we always get the same output. Consequently we have turned to getting an alternate formula that obviously gives only one output.

Define a function $d\colon \mathcal{M}_{n\times n} \to \mathbb{R}$ by altering the permutation expansion formula, replacing $|P_\phi|$ with $\operatorname{sgn}(\phi)$.

$$d(T) = \sum_{\text{permutations } \phi} t_{1,\phi(1)} t_{2,\phi(2)} \cdots t_{n,\phi(n)} \cdot \operatorname{sgn}(\phi)$$

The advantage of this formula is that the number of inversions is clearly well-defined — just count them. Therefore, we will be finished showing that an $n\times n$ determinant function exists when we show that d satisfies the conditions in the definition of a determinant.

4.9 *Lemma* The function d above is a determinant. Hence determinant functions $\det_{n\times n}$ exist for every n.

The book has the proof, which verifies that the definition of determinant's four conditions are satisfied by the function d. We shall here instead go through an illustrative example.

Example This is the d expansion formula for a 3×3 matrix. (To save space on the slides, in this example we write $s(\cdots)$ in place of $\operatorname{sgn}(\cdots)$.)

$$\begin{aligned}
d(\begin{pmatrix} a & b & c \\ e & f & g \\ h & i & j \end{pmatrix}) = afj \cdot s(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) + agi \cdot s(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}) \\
+ bej \cdot s(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}) + cei \cdot s(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}) \\
+ cfh \cdot s(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}) + ceg \cdot s(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}) \qquad (*)
\end{aligned}$$

We illustrate the proof's verification of the four conditions using this generic matrix.

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Condition (4) is easy. In terms of equation $(*)$, in the identity matrix all entries are 0 other than a, f, and j, which are 1. So equation $(*)$ has only one non-zero term, and the sum is 1.

For condition (3) consider the effect of $k\rho_2$ on the generic example matrix. Here is the expansion, following $(*)$.

$$\begin{aligned}
d(\begin{pmatrix} 1 & 2 & 3 \\ 4k & 5k & 6k \\ 7 & 8 & 9 \end{pmatrix}) &= 1(5k)9\cdot s(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) + 1(6k)8\cdot s(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}) \\
&\quad + 2(4k)9\cdot s(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}) + 2(6k)7\cdot s(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}) \\
&\quad + 3(4k)8\cdot s(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}) + 3(5k)7\cdot s(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix})
\end{aligned}$$

The k factors out and the result is k times the d value of the original generic matrix.

Next, condition (2). Contrast the d expansion of these two.

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix} \xrightarrow{\rho_2\leftrightarrow\rho_3} \begin{pmatrix}1&2&3\\7&8&9\\4&5&6\end{pmatrix}$$

The first expansion is

$$1(5)9\cdot s(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix})+1(6)8\cdot s(\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix})+2(4)9\cdot s(\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix})$$
$$+2(6)7\cdot s(\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix})+3(4)8\cdot s(\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix})+3(5)7\cdot s(\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix})$$

while the second is this.

$$1(8)6\cdot s(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix})+1(9)5\cdot s(\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix})+2(7)6\cdot s(\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix})$$
$$+2(9)4\cdot s(\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix})+3(7)5\cdot s(\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix})+3(8)4\cdot s(\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix})$$

The two match term-by-term. For instance, both have terms involving $1 \cdot 5 \cdot 9$ and the associated matrices differ by a row swap. The signum function causes them to differ in sign. The expansions are negatives of each other.

Last, condition (1). By equation $(*)$ the second matrix here

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{10\rho_1+\rho_2} \begin{pmatrix} 1 & 2 & 3 \\ 10+4 & 20+5 & 30+6 \\ 7 & 8 & 9 \end{pmatrix}$$

has this d expansion.

$$\begin{aligned} &1(20+5)9 \cdot s(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) + 1(30+6)8 \cdot s(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}) \\ &\quad + 2(10+4)9 \cdot s(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}) + 2(30+6)7 \cdot s(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}) \\ &\quad + 3(10+4)8 \cdot s(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}) + 3(20+5)7 \cdot s(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}) \end{aligned}$$

Break along the plus signs to get two sums. One is the d expansion of the original generic matrix.

$$1(5)9\cdot s(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix})+1(6)8\cdot s(\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix})+2(4)9\cdot s(\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix})$$
$$+2(6)7\cdot s(\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix})+3(4)8\cdot s(\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix})+3(5)7\cdot s(\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix})$$
$$=d(\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix})$$

The other is this.

$$1(20)9\cdot s(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix})+1(30)8\cdot s(\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix})+2(10)9\cdot s(\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix})$$
$$+2(30)7\cdot s(\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix})+3(10)8\cdot s(\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix})+3(20)7\cdot s(\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix})$$

Factor out 10. What's left, by equation $(*)$, is this d expansion.

$$
\begin{aligned}
&1(2)9 \cdot s(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) + 1(3)8 \cdot s(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}) + 2(1)9 \cdot s(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}) \\
&\quad + 2(3)7 \cdot s(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}) + 3(1)8 \cdot s(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}) + 3(2)7 \cdot s(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}) \\
&\quad = d(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix})
\end{aligned}
$$

This matrix has two identical rows so its d expansion gives 0 (condition (2) shows this, since swapping the two rows changes the sign but doesn't change the matrix). Hence the operation $10\rho_1 + \rho_2$ does not change the value of the function d.

The determinant of the transpose

4.10 *Theorem* The determinant of a matrix equals the determinant of its transpose.

Proof The proof is best understood by doing the general 3×3 case. That the argument applies to the $n\times n$ case will be clear.

Compare the permutation expansion of the matrix T

$$\begin{aligned}\begin{vmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{vmatrix} &= t_{1,1}t_{2,2}t_{3,3}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{1,1}t_{2,3}t_{3,2}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\ &\quad + t_{1,2}t_{2,1}t_{3,3}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{1,2}t_{2,3}t_{3,1}\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\ &\quad + t_{1,3}t_{2,1}t_{3,2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + t_{1,3}t_{2,2}t_{3,1}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}\end{aligned}$$

with the permutation expansion of its transpose.

$$\begin{vmatrix} t_{1,1} & t_{2,1} & t_{3,1} \\ t_{1,2} & t_{2,2} & t_{3,2} \\ t_{1,3} & t_{2,3} & t_{3,3} \end{vmatrix} = t_{1,1}t_{2,2}t_{3,3} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{1,1}t_{3,2}t_{2,3} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$
$$+ t_{2,1}t_{1,2}t_{3,3} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,1}t_{3,2}t_{1,3} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$
$$+ t_{3,1}t_{1,2}t_{2,3} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + t_{3,1}t_{2,2}t_{1,3} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

Compare first the six products of t's. The ones in the expansion of T are the same as the ones in the expansion of the transpose; for instance, $t_{1,2}t_{2,3}t_{3,1}$ is in the top and $t_{3,1}t_{1,2}t_{2,3}$ is in the bottom. That's perfectly sensible — the six in the top arise from all of the ways of picking one entry of T from each row and column while the six in the bottom are all of the ways of picking one entry of T from each column and row, so of course they are the same set.

Next observe that in the two expansions, each t-product expression is not necessarily associated with the same permutation matrix. For instance, on the top $t_{1,2}t_{2,3}t_{3,1}$ is associated with the matrix for the map $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$. On the bottom $t_{3,1}t_{1,2}t_{2,3}$ is associated with the matrix for the map $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 2$. The second map is inverse to the first. This is also perfectly sensible — both the matrix transpose and the map inverse flip the $1, 2$ to $2, 1$, flip the $2, 3$ to $3, 2$, and flip $3, 1$ to $1, 3$.

We finish by noting that the determinant of P_ϕ equals the determinant of $P_{\phi^{-1}}$, as Exercise 16 shows. QED

Example We know the formula for 2×2 matrices.

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad |A^{\mathsf{T}}| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - cb$$