Three.VI Projection

Linear Algebra

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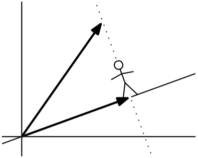
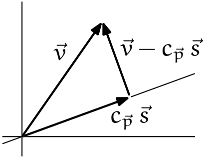
Orthogonal Projection Into a Line

Project a vector into a line

This shows a figure walking out on the line to a point $\vec{p}$ such that the tip of $\vec{v}$ is directly above them, where "above" does not mean parallel to the y-axis but instead means orthogonal to the line.





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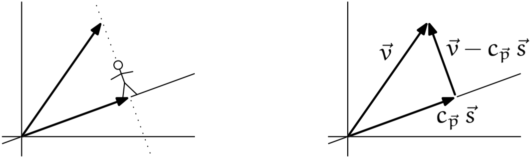

Since the line is the span of some vector $\ell = \{c \cdot \vec{s} \mid c \in \mathbb{R}\}$, we have a coefficient $c_{\vec{p}}$ with the property that $\vec{v} - c_{\vec{p}}\vec{s}$ is orthogonal to $c_{\vec{p}}\vec{s}$.

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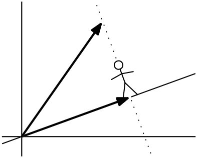

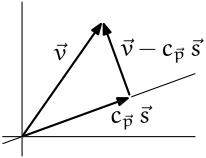

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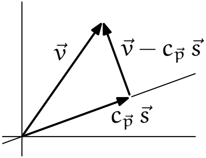
To solve for this coefficient, observe that because $\vec{v} - c_{\vec{p}}\vec{s}$ is orthogonal to a scalar multiple of $\vec{s}$, it must be orthogonal to $\vec{s}$ itself. Then $(\vec{v} - c_{\vec{p}}\vec{s}) \bullet \vec{s} = 0$ gives that $c_{\vec{p}} = \vec{v} \bullet \vec{s} / \vec{s} \bullet \vec{s}$.

We have decomposed $\vec{v}$ into two parts $\vec{v} = (c_{\vec{p}}\vec{s}) + (v - c_{\vec{p}}\vec{s})$.



Intuitively, some of $\vec{v}$ lies with the line and that gives the first part $c_{\vec{p}}\vec{s}$. The part of $\vec{v}$ that lies with a line orthogonal to ℓ is $\vec{v} - c_{\vec{p}}\vec{s}$. What's compelling about pairing these two parts is that they don't interact, in that the projection of one into the line spanned by the other is the zero vector.

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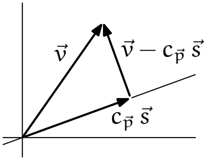

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Note: We have not given a definition of 'angle' in spaces other than $\mathbb{R}^n$'s, so we will stick here to those spaces. Extending the definitions to other spaces is perfectly possible but we don't need them here.

1.1 *Definition* The *orthogonal projection of $\vec{v}$ into the line spanned by a nonzero $\vec{s}$* is this vector.

$$\operatorname{proj}_{[\vec{s}\,]}(\vec{v}) = \frac{\vec{v} \bullet \vec{s}}{\vec{s} \bullet \vec{s}} \cdot \vec{s}$$

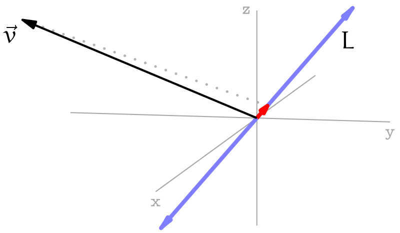
Example The projection of this $\mathbb{R}^3$ vector into the line

$$\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad L = \{c \cdot \vec{s} = c \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \mid c \in \mathbb{R}\}$$

is this vector.

$$\frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/6 \\ 1/6 \end{pmatrix}$$

Because $\vec{v}$ is nearly orthogonal to the line L

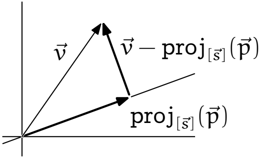


only a small part of $\vec{v}$ lies with the direction of that line, so the projected-to red vector $\mathrm{proj}_{[\vec{s}\,]}(\vec{v})$ is quite short: ($|\vec{v}| = \sqrt{6} \approx 2.45$ while $|\mathrm{proj}_{[\vec{s}\,]}(\vec{v})| = \sqrt{1/6} \approx 0.41$).

Gram-Schmidt Orthogonalization

Mutually orthogonal vectors

The prior subsection suggests that projecting a vector $\vec{v}$ into the line spanned by $\vec{s}$ decomposes $\vec{v}$ into two parts, a part with the line and a part orthogonal to that.



$$\vec{v} = \mathrm{proj}_{[\vec{s}\,]}(\vec{v}) + \left(\vec{v} - \mathrm{proj}_{[\vec{s}\,]}(\vec{v})\right)$$

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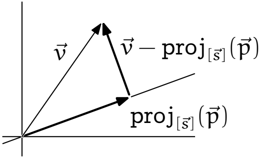

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Because these are orthogonal they are in some sense non-interacting. Here we will develop that.

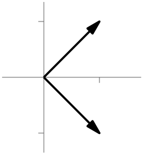
2.1 *Definition* Vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$ are *mutually orthogonal* when any two are orthogonal: if $i \neq j$ then the dot product $\vec{v}_i \cdot \vec{v}_j$ is zero.

Example The vectors of the standard basis $\mathcal{E}_3 \subset \mathbb{R}^3$ are mutually orthogonal.

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Example These two vectors in $\mathbb{R}^2$ are mutually orthogonal.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



The next result makes 'non-interacting' precise.

2.2 *Theorem* If the vectors in a set $\{\vec{v}_1, \ldots, \vec{v}_k\} \subset \mathbb{R}^n$ are mutually orthogonal and nonzero then that set is linearly independent.

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Proof Consider $\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$. For $i \in \{1, .., k\}$, taking the dot product of $\vec{v}_i$ with both sides of the equation $\vec{v}_i \bullet (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) = \vec{v}_i \bullet \vec{0}$, which gives $c_i \cdot (\vec{v}_i \bullet \vec{v}_i) = 0$, shows that $c_i = 0$ since $\vec{v}_i \neq \vec{0}$. QED

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2.5 *Definition* An *orthogonal basis* for a vector space is a basis of mutually orthogonal vectors.

2.7 *Theorem* If $\langle\vec{\beta}_1,\ldots\vec{\beta}_k\rangle$ is a basis for a subspace of $\mathbb{R}^n$ then the vectors

$$\begin{aligned}
\vec{\kappa}_1 &= \vec{\beta}_1 \\
\vec{\kappa}_2 &= \vec{\beta}_2 - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2) \\
\vec{\kappa}_3 &= \vec{\beta}_3 - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \mathrm{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3) \\
&\vdots \\
\vec{\kappa}_k &= \vec{\beta}_k - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_k) - \cdots - \mathrm{proj}_{[\vec{\kappa}_{k-1}]}(\vec{\beta}_k)
\end{aligned}$$

form an orthogonal basis for the same subspace.

The book has the proof. We will instead illustrate.

Example This basis for $\mathbb{R}^2$

$$B = \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \rangle$$

does not have orthogonal vectors. To derive from it a basis $K = \langle \vec{\kappa}_1, \vec{\kappa}_2 \rangle$ that is orthogonal, start by taking the first vector unchanged.

$$\vec{\kappa}_1 = \vec{\beta}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\vec{\kappa}_2$ take the part of $\vec{\beta}_2$ that does not lie with $\vec{\kappa}_1$.

$$\vec{\kappa}_2 = \vec{\beta}_2 - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2/5 \\ 1/5 \end{pmatrix}$$

Note that $\vec{\kappa}_1$ and $\vec{\kappa}_2$ are indeed orthogonal.

Example This is a basis for $\mathbb{R}^3$.

$$B = \langle \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\3\\-1\end{pmatrix} \rangle$$

Start the orthogonal basis with $\vec{\beta}_1$.

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As in the prior slide, the next step is $\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)$.

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}$$

The third step is $\vec{\kappa}_3 = \vec{\beta}_3 - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \mathrm{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)$.

$$\begin{pmatrix}0\\3\\-1\end{pmatrix} - \frac{\begin{pmatrix}0\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}}{\begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix} - \frac{\begin{pmatrix}0\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}-3/2\\3/2\\0\end{pmatrix}}{\begin{pmatrix}-3/2\\3/2\\0\end{pmatrix}\cdot\begin{pmatrix}-3/2\\3/2\\0\end{pmatrix}}\cdot\begin{pmatrix}-3/2\\3/2\\0\end{pmatrix} = \begin{pmatrix}4/3\\4/3\\-4/3\end{pmatrix}$$

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The members of B are at odd angles but the members of K are mutually orthogonal.

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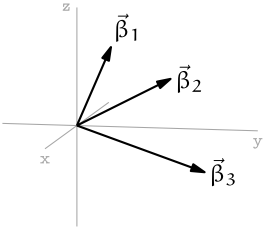

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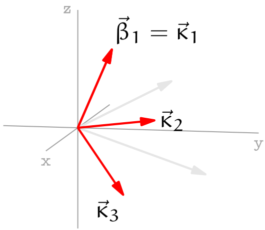
$$K = \langle \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}-3/2\\3/2\\0\end{pmatrix}, \begin{pmatrix}4/3\\4/3\\-4/3\end{pmatrix} \rangle$$



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$$\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}}{\begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 4/3 \\ -4/3 \end{pmatrix}$$

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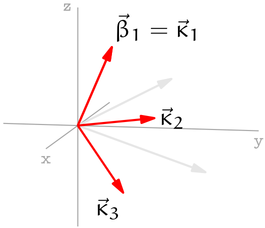

We could go on to make this basis even more like $\mathcal{E}_3$ by normalizing all of its members to have length 1, making an *orthonormal* basis.