Three.V Change of Basis

Linear Algebra

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Changing representations of vectors

Coordinates vary with the basis

Consider this vector $\vec{v} \in \mathbb{R}^3$ and bases for the space.

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathcal{E}_3 = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle \quad B = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rangle$$

With respect to the different bases, the coordinates of $\vec{v}$ are different.

$$\mathrm{Rep}_{\mathcal{E}_3}(\vec{v}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathrm{Rep}_{B}(\vec{v}) = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

In this section we will see how to convert the representation of a vector with respect to a first basis to its representation with respect to a second.

Change of basis matrix

Think of translating from $\mathrm{Rep}_{\mathrm{B}}(\vec{v})$ to $\mathrm{Rep}_{\mathrm{D}}(\vec{v})$ as holding the vector constant. This is the arrow diagram.

$$\begin{array}{c} V_{\mathit{wrt}\ \mathrm{B}} \\ \mathrm{id}\big\downarrow \\ V_{\mathit{wrt}\ \mathrm{D}} \end{array}$$

(This diagram is vertical to fit with the ones in the next subsection.)

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(This diagram is vertical to fit with the ones in the next subsection.)

1.1 *Definition* The *change of basis matrix* for bases $B, D \subset V$ is the representation of the identity map $\mathrm{id}\colon V \to V$ with respect to those bases.

$$\mathrm{Rep}_{B,D}(\mathrm{id}) = \begin{pmatrix} \vdots & & \vdots \\ \mathrm{Rep}_D(\vec{\beta}_1) & \cdots & \mathrm{Rep}_D(\vec{\beta}_n) \\ \vdots & & \vdots \end{pmatrix}$$

1.3 *Lemma* To convert from the representation of a vector $\vec{v}$ with respect to B to its representation with respect to D use the change of basis matrix.

$$\mathrm{Rep}_{B,D}(\mathrm{id})\,\mathrm{Rep}_{B}(\vec{v}) = \mathrm{Rep}_{D}(\vec{v})$$

Conversely, if left-multiplication by a matrix changes bases $M \cdot \mathrm{Rep}_{B}(\vec{v}) = \mathrm{Rep}_{D}(\vec{v})$ then M is a change of basis matrix.

The book has the proof.

Example To change a representation of a member of $\mathcal{P}_2$ from being with respect to $B = \langle 1, 1+x, 1+x+x^2 \rangle$ to being with respect to $D = \langle x^2-1, x, x^2+1 \rangle$, compute $\text{Rep}_{B,D}(\text{id})$. The identity map acting on the elements of B has no effect. Represent those elements with respect to D.

$$\text{Rep}_D(1) = \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix} \quad \text{Rep}_D(1+x) = \begin{pmatrix} -1/2 \\ 1 \\ 1/2 \end{pmatrix} \quad \text{Rep}_D(1+x+x^2) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The change of basis matrix is the concatenation of those.

$$\text{Rep}_{B,D}(\text{id}) = \begin{pmatrix} -1/2 & -1/2 & 0 \\ 0 & 1 & 1 \\ 1/2 & 1/2 & 1 \end{pmatrix}$$

For example, we can translate the representation of $\vec{v} = 2 - x + 3x^2$.

$$\text{Rep}_B(\vec{v}) = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} \qquad \text{Rep}_D(\vec{v}) = \begin{pmatrix} 1/2 \\ -1 \\ 5/2 \end{pmatrix} = \begin{pmatrix} -1/2 & -1/2 & 0 \\ 0 & 1 & 1 \\ 1/2 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$$

1.5 *Lemma* A matrix changes bases if and only if it is nonsingular.

The book contains the proof; we will do an example.

Example Recall that any nonsingular matrix M decomposes into a product of elementary reduction matrices $M = R_1 \cdots R_r$. We can show that M changes bases by showing that each R_i changes bases. Consider this 3×3 case.

$$C_{1,3}(-4) \cdot \mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ -4r_1 + r_3 \end{pmatrix}$$

Where $B = \langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \rangle$, the right side represents the same vector $\vec{v}$ with respect to $\hat{B} = \langle \vec{\beta}_1 + 4\vec{\beta}_3, \vec{\beta}_2, \vec{\beta}_3 \rangle$.

$$r_1 \cdot (\vec{\beta}_1 + 4\vec{\beta}_3) + r_2 \cdot \beta_2 + (-4r_1 + r_3) \cdot \vec{\beta}_3 = r_1\vec{\beta}_1 + r_2\beta_2 + r_3\vec{\beta}_3$$

Verifying that $\hat{B}$ is a basis is routine.

1.6 *Corollary* A matrix is nonsingular if and only if it represents the identity map with respect to some pair of bases.

Changing map representations

The natural next step is to see how to convert $\mathrm{Rep}_{B,D}(h)$ to $\mathrm{Rep}_{\hat{B},\hat{D}}(h)$. Here is the arrow diagram.

$$\begin{array}{ccc}
V_{\mathit{wrt}\ B} & \xrightarrow[H]{h} & W_{\mathit{wrt}\ D} \\
\mathrm{id}\big\downarrow & & \mathrm{id}\big\downarrow \\
V_{\mathit{wrt}\ \hat{B}} & \xrightarrow[\hat{H}]{h} & W_{\mathit{wrt}\ \hat{D}}
\end{array}$$

To move from the lower-left to the lower-right we can either go straight over, or else up to V_B then over to W_D and then down. So we can calculate $\hat{H} = \mathrm{Rep}_{\hat{B},\hat{D}}(h)$ either by directly using $\hat{B}$ and $\hat{D}$, or else by first changing bases with $\mathrm{Rep}_{\hat{B},B}(\mathrm{id})$ then multiplying by $H = \mathrm{Rep}_{B,D}(h)$ and then changing bases with $\mathrm{Rep}_{D,\hat{D}}(\mathrm{id})$.

Theorem To convert from the matrix H representing a map h with respect to B, D to the matrix $\hat{H}$ representing it with respect to $\hat{B}, \hat{D}$ use this formula.

$$\hat{H} = \mathrm{Rep}_{D,\hat{D}}(\mathrm{id}) \cdot H \cdot \mathrm{Rep}_{\hat{B},B}(\mathrm{id}) \qquad (*)$$

Proof This is evident from the diagram. QED

Example Consider the derivative map $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_2$ as well as $B = \langle 1, 1+x, 1+x+x^2 \rangle, D = \langle 1+x^2, x, 1-x^2 \rangle$ and $\hat{B} = \langle 1, x, x^2 \rangle, \hat{D} = \langle 1+x, x+x^2, 1+x^2 \rangle$.

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We can find H and $\hat{H}$ using the methods we have already seen.

$$\mathrm{Rep}_{B,D}(d/dx) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \quad \mathrm{Rep}_{\hat{B},\hat{D}}(d/dx) = \begin{pmatrix} 0 & 1/2 & 1 \\ 0 & -1/2 & 1 \\ 0 & 1/2 & -1 \end{pmatrix}$$

These do the base changes.

$$\mathrm{Rep}_{\hat{B},B}(\mathrm{id}) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathrm{Rep}_{D,\hat{D}}(\mathrm{id}) = \begin{pmatrix} 0 & 1/2 & 1 \\ 0 & 1/2 & -1 \\ 1 & -1/2 & 0 \end{pmatrix}$$

We can check this case of equation $(*)$ by multiplying through.

$$\begin{aligned}
\mathrm{Rep}_{\hat{B},\hat{D}}(d/dx) &= \mathrm{Rep}_{D,\hat{D}}(\mathrm{id}) \cdot \mathrm{Rep}_{B,D}(d/dx) \cdot \mathrm{Rep}_{\hat{B},B}(\mathrm{id}) \\
&= \begin{pmatrix} 0 & 1/2 & 1 \\ 0 & 1/2 & -1 \\ 1 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Example The map $t_{\pi/6} \colon \mathbb{R}^2 \to \mathbb{R}^2$ rotating vectors counterclockwise by $\pi/6$ radians has this representation with respect to the standard bases.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t_{\pi/6}) = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

We can translate to the representatation with respect to

$$B = D = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle$$

using the change of basis matrices. Here is the diagram, specialized for this case.

$$\begin{tikzcd}
\mathbb{R}^2_{wrt\ \mathcal{E}_2} \arrow[r, "t_{\pi/6}", "\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t_{\pi/6})"'] \arrow[d, "\mathrm{id}"'] & \mathbb{R}^2_{wrt\ \mathcal{E}_2} \arrow[d, "\mathrm{id}"'] \\
\mathbb{R}^2_{wrt\ B} \arrow[r, "t_{\pi/6}", "\hat{H}"'] & \mathbb{R}^2_{wrt\ D}
\end{tikzcd}$$

To get $\hat{H}$ we move up from the upper left, across, and then down.

With respect to the standard basis real vectors represent themselves, so the matrix representing moving up is easy.

$$\mathrm{Rep}_{B,\mathcal{E}_2}(\mathrm{id}) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The matrix for moving down is the inverse of the prior one.

$$\mathrm{Rep}_{\mathcal{E}_2,D}(\mathrm{id}) = (\mathrm{Rep}_{D,\mathcal{E}_2}(\mathrm{id}))^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{-1}{2} \cdot \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

Thus we have this.

$$\begin{aligned} \mathrm{Rep}_{B,D}(t_{\pi/6}) &= \frac{-1}{2} \cdot \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} \end{aligned}$$

2.4 *Definition* Same-sized matrices H and $\hat{H}$ are *matrix equivalent* if there are nonsingular matrices P and Q such that $\hat{H} = PHQ$.

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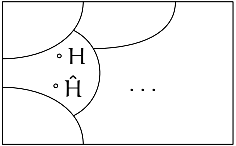
Exercise 24 checks that matrix equivalence is an equivalence relation. Thus it partitions the set of matrices into matrix equivalence classes.

All matrices:



H matrix equivalent to $\hat{H}$

Canonical form for matrix equivalence

2.7 *Theorem* Any $m \times n$ matrix of rank k is matrix equivalent to the $m \times n$ matrix that is all zeros except that the first k diagonal entries are ones.

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ & \vdots & & & & & \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ & \vdots & & & & & \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

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This is a *block partial-identity* form.

$$\left(\begin{array}{c|c} I & Z \\ \hline Z & Z \end{array}\right)$$

For an example, recall first that any nonsingular matrix M can be factored into a product $M = R_1 \cdots R_r$ where each R_t is an elementary reduction matrix $C_{i,j}(k)$, or $M_i(k)$, or $P_{i,j}$.

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Recall also that from the left matrices operate on rows

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 0 & -3 \end{pmatrix}$$

while from the right they act on columns: this does $-4 \cdot \text{col}_3 + \text{col}_1$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -11 & 2 & 3 \\ -20 & 5 & 6 \\ -29 & 8 & 9 \end{pmatrix}$$

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Example This matrix has rank 2.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

We will find P and Q to get the 3×3 canonical matrix of rank 2.

Row operations produce echelon form.

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\xrightarrow[-7\rho_1+\rho_3]{-4\rho_1+\rho_2}\xrightarrow{-2\rho_2+\rho_3}\xrightarrow{-(1/3)\rho_2}\begin{pmatrix}1&2&3\\0&1&2\\0&0&0\end{pmatrix}$$

Multiplying the reduction matrices gives P; note the right to left order.

$$\begin{pmatrix}1&0&0\\0&-1/3&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&-2&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-7&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-4&1&0\\0&0&1\end{pmatrix}$$
$$=\begin{pmatrix}1&0&0\\4/3&-1/3&0\\1&-2&1\end{pmatrix}$$

So far we have this.

$$PHQ=\begin{pmatrix}1&0&0\\4/3&-1/3&0\\1&-2&1\end{pmatrix}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}Q=\begin{pmatrix}1&2&3\\0&1&2\\0&0&0\end{pmatrix}Q$$

Column operations produce the block partial identity.

$$\begin{pmatrix}1&2&3\\0&1&2\\0&0&0\end{pmatrix} \xrightarrow{-2\mathrm{col}_2+\mathrm{col}_3} \xrightarrow{-2\mathrm{col}_1+\mathrm{col}_2} \xrightarrow{\mathrm{col}_1+\mathrm{col}_3} \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

Combine the reduction matrices to get Q. In contrast with the construction of P, here they come left to right.

$$\begin{pmatrix}1&0&0\\0&1&-2\\0&0&1\end{pmatrix}\begin{pmatrix}1&-2&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&-2&1\\0&1&-2\\0&0&1\end{pmatrix}$$

In sum, we have this equation.

$$PHQ=\begin{pmatrix}1&0&0\\4/3&-1/3&0\\1&-2&1\end{pmatrix}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}1&-2&1\\0&1&-2\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

Action of a canonical form matrix

This kind of matrix is has an easy to understand effect.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

It is a projection — the map $t \colon \mathbb{R}^3 \to \mathbb{R}^3$ represented with respect to the standard bases by this block partial identity matrix is projection from three space to the xy plane.

Matrix equivalence is characterized by rank

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Proof Two same-sized matrices with the same rank are equivalent to the same block partial-identity matrix. QED