Three.II Homomorphisms

Linear Algebra, edition four

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Definition

Homomorphism

1.1 *Definition* A function between vector spaces $h\colon V \to W$ that preserves addition

$$\text{if } \vec{v}_1, \vec{v}_2 \in V \text{ then } h(\vec{v}_1 + \vec{v}_2) = h(\vec{v}_1) + h(\vec{v}_2)$$

and scalar multiplication

$$\text{if } \vec{v} \in V \text{ and } r \in \mathbb{R} \text{ then } h(r \cdot \vec{v}) = r \cdot h(\vec{v})$$

is a *homomorphism* or *linear map*.

Example Of these two maps $h, g \colon \mathbb{R}^2 \to \mathbb{R}$, the first is a homomorphism while the second is not.

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{h} 2x - 3y \qquad \begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{g} 2x - 3y + 1$$

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The map h respects addition

$$\begin{aligned} h(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) = h(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}) &= 2(x_1 + x_2) - 3(y_1 + y_2) \\ &= (2x_1 - 3y_1) + (2x_2 - 3y_2) = h(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}) + h(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) \end{aligned}$$

and scalar multiplication.

$$r \cdot h(\begin{pmatrix} x \\ y \end{pmatrix}) = r \cdot (2x - 3y) = 2rx - 3ry = (2r)x - (3r)y = h(r \cdot \begin{pmatrix} x \\ y \end{pmatrix})$$

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In contrast, g does not respect addition.

$$g(\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}) = -17 \qquad g(\begin{pmatrix} 1 \\ 4 \end{pmatrix}) + g(\begin{pmatrix} 5 \\ 6 \end{pmatrix}) = -16$$

We proved these two while studying isomorphisms.

1.6 *Lemma* A linear map sends the zero vector to the zero vector.

1.7 *Lemma* The following are equivalent for any map $f\colon V \to W$ between vector spaces.

(1) f is a homomorphism
(2) $f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)$ for any $c_1, c_2 \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2 \in V$
(3) $f(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n) = c_1 \cdot f(\vec{v}_1) + \cdots + c_n \cdot f(\vec{v}_n)$ for any $c_1, \ldots, c_n \in \mathbb{R}$ and $\vec{v}_1, \ldots, \vec{v}_n \in V$

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To verify that a map is a homomorphism, we most often use (2).

Example Between any two vector spaces the zero map $Z\colon V \to W$ given by $Z(\vec{v}) = \vec{0}_W$ is a linear map. Using (2): $Z(c_1\vec{v}_1 + c_2\vec{v}_2) = \vec{0}_W = \vec{0}_W + \vec{0}_W = c_1 Z(\vec{v}_1) + c_2 Z(\vec{v}_2)$.

Example The *inclusion map* $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$

$$\iota(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

is a homomorphism.

$$\begin{aligned}
\iota(c_1 \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) &= \iota(\begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{pmatrix}) \\
&= \begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} c_1x_1 \\ c_1y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} c_2x_2 \\ c_2y_2 \\ 0 \end{pmatrix} \\
&= c_1 \cdot \iota(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}) + c_2 \cdot \iota(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix})
\end{aligned}$$

Example The derivative is a transformation on polynomial spaces. For instance, consider $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_1$ given by

$$d/dx\,(ax^2 + bx + c) = 2ax + b$$

(examples are $d/dx\,(3x^2 - 2x + 4) = 6x - 2$ and $d/dx\,(x^2 + 1) = 2x$).

It is a homomorphism.

$$\begin{aligned}
d/dx\,\bigl(\,r_1(a_1x^2 + b_1x + c_1) + r_2(a_2x^2 + b_2x + c_2)\,\bigr) \\
&= d/dx\,\bigl(\,(r_1a_1 + r_2a_2)x^2 + (r_1b_1 + r_2b_2)x + (r_1c_1 + r_2c_2)\,\bigr) \\
&= 2(r_1a_1 + r_2a_2)x + (r_1b_1 + r_2b_2) \\
&= (2r_1a_1x + r_1b_1) + (2r_2a_2x + r_2b_2) \\
&= r_1 \cdot d/dx\,(a_1x^2 + b_1x + c_1) + r_2 \cdot d/dx\,(a_2x^2 + b_2x + c_2)
\end{aligned}$$

Example The *trace* of a square matrix is the sum down the upper-left to lower-right diagonal. Thus $\mathrm{Tr}\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ is this.

$$\mathrm{Tr}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a + d$$

It is linear.

$$\begin{aligned}
\mathrm{Tr}(\, r_1 \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\,) \\
&= \mathrm{Tr}(\begin{pmatrix} r_1 a_1 + r_2 a_2 & r_1 b_1 + r_2 b_2 \\ r_1 c_1 + r_2 c_2 & r_1 d_1 + r_2 d_2 \end{pmatrix}) \\
&= (r_1 a_1 + r_2 a_2) + (r_1 d_1 + r_2 d_2) \\
&= r_1 (a_1 + d_1) + r_2 (a_2 + d_2) \\
&= r_1 \cdot \mathrm{Tr}(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) + r_2 \cdot \mathrm{Tr}(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})
\end{aligned}$$

1.9 *Theorem* A homomorphism is determined by its action on a basis: if V is a vector space with basis $\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$, if W is a vector space, and if $\vec{w}_1,\ldots,\vec{w}_n\in W$ (these codomain elements need not be distinct) then there exists a homomorphism from V to W sending each $\vec{\beta}_i$ to $\vec{w}_i$, and that homomorphism is unique.

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Example The book has the proof. Here is an illustration. Consider a map $h\colon\mathbb{R}^2\to\mathbb{R}^2$ with this action on a basis.

$$\begin{pmatrix}1\\1\end{pmatrix}\xmapsto{h}\begin{pmatrix}1\\3\end{pmatrix}\qquad\begin{pmatrix}1\\0\end{pmatrix}\xmapsto{h}\begin{pmatrix}2\\0\end{pmatrix}$$

The effect of the map on any vector $\vec{v}$ at all is determined by those two facts. Represent that vector $\vec{v}$ with respect to the basis.

$$\begin{pmatrix}-1\\5\end{pmatrix}=5\cdot\begin{pmatrix}1\\1\end{pmatrix}-6\cdot\begin{pmatrix}1\\0\end{pmatrix}$$

Compute $h(\vec{v})$ using the definition of homomorphism.

$$h(\vec{v})=h(\,5\cdot\begin{pmatrix}1\\1\end{pmatrix}-6\cdot\begin{pmatrix}1\\0\end{pmatrix}\,)=5\cdot\begin{pmatrix}1\\3\end{pmatrix}-6\cdot\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}-7\\15\end{pmatrix}$$

Example Consider $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ with this effect on the standard basis.

$$f(\vec{e}_1) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad f(\vec{e}_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad f(\vec{e}_3) = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

Because this is the standard basis, the effect of the map on any vector $\vec{v} \in \mathbb{R}^3$ is especially easy to compute. For instance,

$$\mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(\begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix}) = \begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix}$$

and so we have this.

$$f(\begin{pmatrix} -5 \\ 0 \\ 10 \end{pmatrix}) = -5 \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 10 \cdot \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ -20 \\ 5 \end{pmatrix}$$

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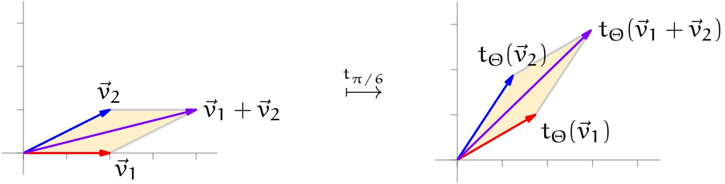
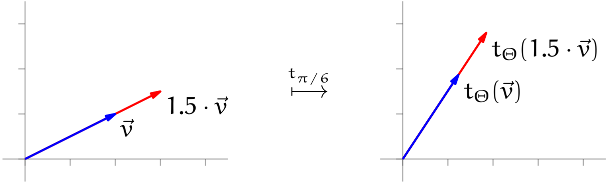
1.10 *Definition* Let V and W be vector spaces and let $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ be a basis for V. A function defined on that basis $f\colon B \to W$ is *extended linearly* to a function $\hat{f}\colon V \to W$ if for all $\vec{v} \in V$ such that $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$, the action of the map is $\hat{f}(\vec{v}) = c_1 \cdot f(\vec{\beta}_1) + \cdots + c_n \cdot f(\vec{\beta}_n)$.

Example Consider the action $t_\Theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ of rotating all vectors in the plane through an angle Θ. These drawings show that this map satisfies the addition

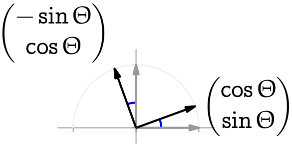


and scalar multiplication conditions.



We will develop the formula for t_Θ.

Fix a basis for the domain $\mathbb{R}^2$; the standard basis $\mathcal{E}_2$ is convenient. We want the basis vectors mapped as here.



$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

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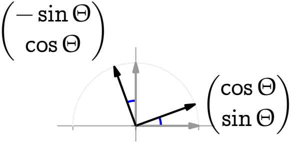

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Extend linearly.

$$\begin{aligned}
t_\theta(\begin{pmatrix} x \\ y \end{pmatrix}) &= t_\theta(x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \\
&= x \cdot t_\theta(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) + y \cdot t_\theta(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) \\
&= x \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + y \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}
\end{aligned}$$

Example One basis of the space of quadratic polynomials $\mathcal{P}_2$ is $B = \langle x^2, x, 1 \rangle$. Define the *evaluation map* $\text{eval}_3 \colon \mathcal{P}_2 \to \mathbb{R}$ by specifying its action on that basis

$$x^2 \xmapsto{\text{eval}_3} 9 \quad x \xmapsto{\text{eval}_3} 3 \quad 1 \xmapsto{\text{eval}_3} 1$$

and then extending linearly.

$$\begin{aligned}\text{eval}_3(ax^2 + bx + c) &= a \cdot \text{eval}_3(x^2) + b \cdot \text{eval}_3(x) + c \cdot \text{eval}_3(1) \\ &= 9a + 3b + c\end{aligned}$$

For instance, $\text{eval}_3(x^2 + 2x + 3) = 9 + 6 + 3 = 18$.

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For instance, $\text{eval}_3(x^2 + 2x + 3) = 9 + 6 + 3 = 18$.

On the basis elements, we can describe the action of this map as: plugging the value 3 in for x. That remains true when we extend linearly, so $\text{eval}_3(\,p(x)\,) = p(3)$.

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Example In $\mathbb{R}^3$ the function f_{yz} that reflects vectors over the yz-plane

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\stackrel{f_{yz}}{\longmapsto}\begin{pmatrix}-x\\y\\z\end{pmatrix}$$

is a linear transformation.

$$\begin{aligned}f_{yz}(r_1\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+r_2\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix})&=f_{yz}(\begin{pmatrix}r_1x_1+r_2x_2\\r_1y_1+r_2y_2\\r_1z_1+r_2z_2\end{pmatrix})=\begin{pmatrix}-(r_1x_1+r_2x_2)\\r_1y_1+r_2y_2\\r_1z_1+r_2z_2\end{pmatrix}\\&=r_1\begin{pmatrix}-x_1\\y_1\\z_1\end{pmatrix}+r_2\begin{pmatrix}-x_2\\y_2\\z_2\end{pmatrix}=r_1f_{yz}(\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix})+r_2f_{yz}(\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix})\end{aligned}$$

1.17 *Lemma* For vector spaces V and W, the set of linear functions from V to W is itself a vector space, a subspace of the space of all functions from V to W.

We denote the space of linear maps from V to W by $\mathcal{L}(V, W)$.

The book contains the proof; instead we will do a couple of examples.

Example We can combine the two homomorphisms $f, g\colon \mathcal{P}_1 \to \mathbb{R}^2$

$$f(a_0 + a_1 x) = \begin{pmatrix} a_0 + a_1 \\ 0 \end{pmatrix} \qquad g(a_0 + a_1 x) = \begin{pmatrix} 4a_1 \\ a_1 \end{pmatrix}$$

into a function $2f + 3g\colon \mathcal{P}_1 \to \mathbb{R}^2$ whose action is this.

$$(2f + 3g)\,(a_0 + a_1 x) = \begin{pmatrix} 2a_0 + 14a_1 \\ 3a_1 \end{pmatrix}$$

The point of the lemma is that the linear combination $2f + 3g$ is also a homomorphism; the check is routine. The collection of homomorphisms from $\mathcal{P}_1$ to $\mathbb{R}^2$ is closed under linear combinations of those homomorphisms — it is a vector space.

Example Consider $\mathcal{L}(\mathbb{R},\mathbb{R}^2)$. A member of $\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ is a linear map. A linear map is determined by its action on a basis of the domain space. Fix these bases.

$$B_{\mathbb{R}} = \mathcal{E}_1 = \langle 1 \rangle \qquad B_{\mathbb{R}^2} = \mathcal{E}_2 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$$

Thus the functions that are elements of $\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ are determined by c_1 and c_2 here.

$$1 \stackrel{t}{\longmapsto} c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We could write each such map as $h = h_{c_1,c_2}$. There are two parameters and thus $\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ is a dimension 2 space.

Range space and null space

2.1 *Lemma* Under a homomorphism, the image of any subspace of the domain is a subspace of the codomain. In particular, the image of the entire space, the range of the homomorphism, is a subspace of the codomain.

The book has the proof; we instead consider an example.

Example Let $f \colon \mathbb{R}^2 \to \mathcal{M}_{2\times 2}$ be

$$\begin{pmatrix} a \\ b \end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix} a & a+b \\ 2b & b \end{pmatrix}$$

(the check that it is a homomorphism is routine). One subspace of the domain is the x axis.

$$S = \{ \begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in \mathbb{R} \}$$

The image under f of the x axis is a subspace of of the codomain $\mathcal{M}_{2\times 2}$.

$$f(S) = \{ \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \}$$

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Another subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$ itself. The image of $\mathbb{R}^2$ under f is this subspace of $\mathcal{M}_{2\times 2}$.

$$f(\mathbb{R}^2) = \{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot c_1 + \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \cdot c_2 \mid c_1, c_2 \in \mathbb{R} \}$$

Example For any angle θ, the function $t_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors counterclockwise through an angle θ is a homomorphism.

In the domain $\mathbb{R}^2$ each line through the origin is a subspace. The image of that line under this map is another line through the origin, a subspace of the codomain $\mathbb{R}^2$.

Range space

2.2 *Definition* The *range space* of a homomorphism $h \colon V \to W$ is

$$\mathscr{R}(h) = \{ h(\vec{v}) \mid \vec{v} \in V \}$$

sometimes denoted $h(V)$. The dimension of the range space is the map's *rank*.

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Example This map from $\mathcal{M}_{2\times 2}$ to $\mathbb{R}^2$ is linear.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{h}{\longmapsto} \begin{pmatrix} a+b \\ 2a+2b \end{pmatrix}$$

The range space is a line through the origin.

$$\{\begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R}\}$$

Every member of that set is the image of a 2×2 matrix.

$$\begin{pmatrix} t \\ 2t \end{pmatrix} = h(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix})$$

The map's rank is 1.

Example The derivative map $d/dx\colon \mathcal{P}_4 \to \mathcal{P}_4$ is linear. Its range is $\mathscr{R}(d/dx) = \mathcal{P}_3$. (Verifying that every member of $\mathcal{P}_3$ is the derivative of some member of $\mathcal{P}_4$ is easy.) The rank of this derivative function is the dimension of $\mathcal{P}_3$, namely 4.

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Example Projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

is a linear map; the check is routine. The range space is $\mathscr{R}(\pi) = \mathbb{R}^2$ because given a vector $\vec{w} \in \mathbb{R}^2$

$$\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}$$

we can find a $\vec{v} \in \mathbb{R}^3$ that maps to it, specifically any $\vec{v}$ with a first component a and second component b. Thus the rank of π is 2.

Example The derivative map $d/dx\colon \mathcal{P}_4 \to \mathcal{P}_4$ is linear. Its range is $\mathscr{R}(d/dx) = \mathcal{P}_3$. (Verifying that every member of $\mathcal{P}_3$ is the derivative of some member of $\mathcal{P}_4$ is easy.) The rank of this derivative function is the dimension of $\mathcal{P}_3$, namely 4.

Example Projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

is a linear map; the check is routine. The range space is $\mathscr{R}(\pi) = \mathbb{R}^2$ because given a vector $\vec{w} \in \mathbb{R}^2$

$$\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}$$

we can find a $\vec{v} \in \mathbb{R}^3$ that maps to it, specifically any $\vec{v}$ with a first component a and second component b. Thus the rank of π is 2.

In the book's next section, on computing linear maps, we will do more examples of determining the range space.

Many-to-one

In moving from isomorphisms to homomorphisms we dropped the requirement that the maps be onto and one-to-one. But any homomorphism $h\colon V \to W$ is onto its range space $\mathscr{R}(h)$, so dropping the onto condition has, in a way, no effect on the range. It doesn't allow any essentially new maps.

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In contrast, consider the effect of dropping the one-to-one condition. With that, an output vector $\vec{w} \in W$ may have many associated inputs, many $\vec{v} \in V$ such that $h(\vec{v}) = \vec{w}$.

Recall that for any function $h\colon V \to W$, the set of elements of V that map to $\vec{w} \in W$ is the *inverse image* $h^{-1}(\vec{w}) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{w}\}$.

The structure of the inverse image sets will give us insight into the definition of homomorphism.

Example Projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ onto the x axis is linear.

$$\pi(\begin{pmatrix} x \\ y \end{pmatrix}) = x$$

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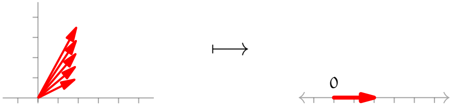
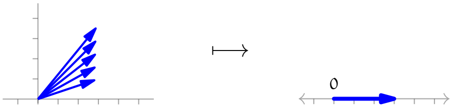
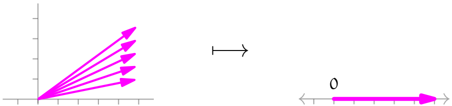
Here are some elements of $\pi^{-1}(2)$. Think of these as "2 vectors."



Think of elements of $\pi^{-1}(3)$ as "3 vectors."



These elements of $\pi^{-1}(5)$ are "5 vectors."



These drawings give us a way to make the definition of homomorphism more concrete. Consider preservation of addition.

$$\pi(\vec{u}) + \pi(\vec{v}) = \pi(\vec{u} + \vec{v})$$

If $\vec{u}$ is such that $\pi(\vec{u}) = 2$, and $\vec{v}$ is such that $\pi(\vec{v}) = 3$, then $\vec{u} + \vec{v}$ will be such that the sum $\pi(\vec{u} + \vec{v}) = 5$.

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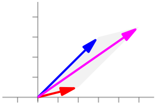
If $\vec{u}$ is such that $\pi(\vec{u}) = 2$, and $\vec{v}$ is such that $\pi(\vec{v}) = 3$, then $\vec{u} + \vec{v}$ will be such that the sum $\pi(\vec{u} + \vec{v}) = 5$. That is, a "2 vector" plus a "3 vector" is a "5 vector." Red plus blue makes magenta.

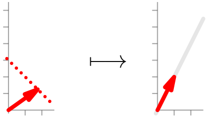


A similar interpretation holds for preservation of scalar multiplication: the image of an "$r \cdot 2$ vector" is r times 2.

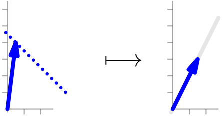
Example This function $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear.

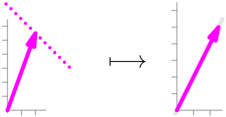
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ 2x+2y \end{pmatrix}$$

Here are elements of $h^{-1}(\begin{pmatrix} 1 \\ 2 \end{pmatrix})$. (Only one inverse image element is shown as a vector, most are indicated with dots.)



Here are some elements of $h^{-1}(\begin{pmatrix} 1.5 \\ 3 \end{pmatrix})$ and $h^{-1}(\begin{pmatrix} 2.5 \\ 5 \end{pmatrix})$.

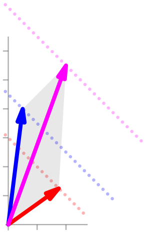




The way that the range space vectors add

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 5 \end{pmatrix}$$

is reflected in the domain: red plus blue makes magenta.



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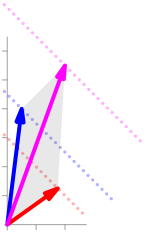
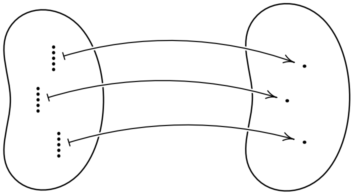


That is, preservation of addition is: $h(\vec{v}_1) + h(\vec{v}_2) = h(\vec{v}_1 + \vec{v}_2)$.

Homomorphisms organize the domain

So the intuition is that a linear map organizes its domain into inverse images,

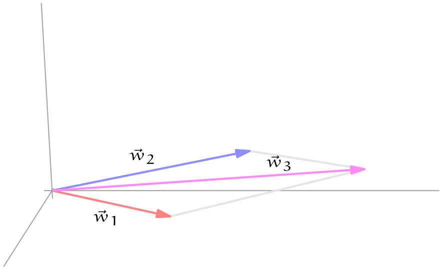


such that those sets reflect the structure of the range.

Example Projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a homomorphism.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

Here we draw the range $\mathbb{R}^2$ as the xy-plane inside of $\mathbb{R}^3$.



In the range the parallelogram shows a vector addition $\vec{w}_1 + \vec{w}_2 = \vec{w}_3$.

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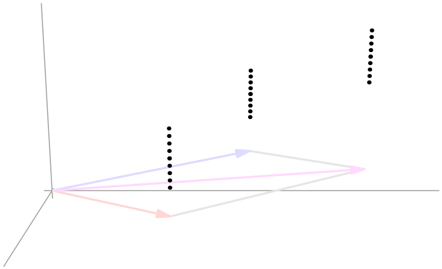

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The diagram shows some of the points in each inverse image $\pi^{-1}(\vec{w}_1)$, $\pi^{-1}(\vec{w}_2)$, and $\pi^{-1}(\vec{w}_3)$.

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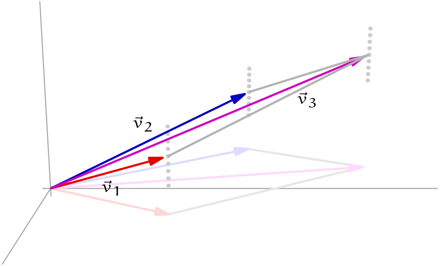

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The diagram shows some of the points in each inverse image $\pi^{-1}(\vec{w}_1)$, $\pi^{-1}(\vec{w}_2)$, and $\pi^{-1}(\vec{w}_3)$. The sum of a vector $\vec{v}_1 \in \pi^{-1}(\vec{w}_1)$ and a vector $\vec{v}_2 \in \pi^{-1}(\vec{w}_2)$ equals a vector $\vec{v}_3 \in \pi^{-1}(\vec{w}_3)$. A $\vec{w}_1$ vector plus a $\vec{w}_2$ vector equals a $\vec{w}_3$ vector.

This interpretation of the definition of homomorphism also holds when the spaces are not ones that we can sketch.

Example Let $h\colon \mathcal{P}_2 \to \mathbb{R}^2$ be

$$ax^2 + bx + c \mapsto \begin{pmatrix} b \\ b \end{pmatrix}$$

and consider these three members of the range such that $\vec{w}_1 + \vec{w}_2 = \vec{w}_3$

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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The inverse image of $\vec{w}_1$ is $h^{-1}(\vec{w}_1) = \{a_1x^2 + 1x + c_1 \mid a_1, c_1 \in \mathbb{R}^2\}$. Members of this set are "$\vec{w}_1$ vectors."

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Any $\vec{v}_1 \in h^{-1}(\vec{w}_1)$ plus any $\vec{v}_2 \in h^{-1}(\vec{w}_2)$ equals a $\vec{v}_3 \in h^{-1}(\vec{w}_3)$: a quadratic with an x coefficient of 1 plus a quadratic with an x coefficient of -1 equals a quadratic with an x coefficient of 0.

Null space

In each of those examples, the homomorphism $h\colon V \to W$ shows how to view the domain V as organized into the inverse images $h^{-1}(\vec{w})$.

In the examples these inverse images are all the same, but shifted. So if we describe one of them then we understand how the domain is divided. Vector spaces have a distinguished element, $\vec{0}$. So we next consider the inverse image $h^{-1}(\vec{0})$.

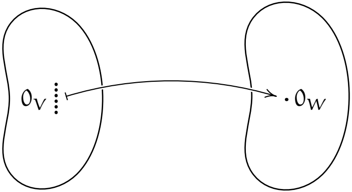
2.10 *Lemma* For any homomorphism the inverse image of a subspace of the range is a subspace of the domain. In particular, the inverse image of the trivial subspace of the range is a subspace of the domain.

The book has the verification.

2.11 *Definition* The *null space* or *kernel* of a linear map $h\colon V \to W$ is the inverse image of $\vec{0}_W$.

$$\mathscr{N}(h) = h^{-1}(\vec{0}_W) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{0}_W\}$$

The dimension of the null space is the map's *nullity*.



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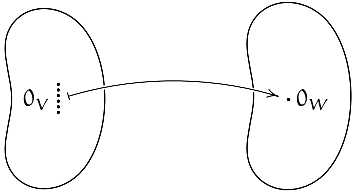


Note Strictly, the trivial subspace of the codomain is not $\vec{0}_W$, it is $\{\vec{0}_W\}$, and so we may think to write the nullspace as $h^{-1}(\{\vec{0}_W\})$. But we have defined the two sets $h^{-1}(\vec{w})$ and $h^{-1}(\{\vec{w}\})$ to be equal and the first is easier to write.

Example Consider the derivative $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_1$. This is the nullspace; note that it is a subset of the domain

$$\mathscr{N}(d/dx) = \{ax^2 + bx + c \mid 2ax + b = 0\}$$

(the '0' there is the zero polynomial $0x + 0$). Now, $2ax + b = 0$ if and only if they have the same constant coefficient $b = 0$, the same x coefficient of $a = 0$, and the same coefficient of x^2 (which gives no restriction). So this is the nullspace, and the nullity is 1.

$$\mathscr{N}(d/dx) = \{ax^2 + bx + c \mid a = 0,\ b = 0,\ c \in \mathbb{R}\} = \{c \mid c \in \mathbb{R}\}$$

Example The function $h\colon \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto 2a + b$$

has this null space and so its nullity is 1.

$$\mathscr{N}(h) = \{\begin{pmatrix} a \\ b \end{pmatrix} \mid 2a + b = 0\} = \{\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} b \mid b \in \mathbb{R}\}$$

Example The homomorphism $f\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix} a+b \\ c+d \end{pmatrix}$$

has this null space

$$\begin{aligned} \mathscr{N}(f) &= \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a+b=0 \text{ and } c+d=0 \} \\ &= \{ \begin{pmatrix} -b & b \\ -d & d \end{pmatrix} \mid b, d \in \mathbb{R} \} \end{aligned}$$

and a nullity of 2.

Example The dilation function $d_3\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 3b \end{pmatrix}$$

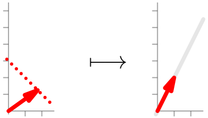
has $\mathscr{N}(d_3) = \{\vec{0}\}$. A trivial space has an empty basis so d_3's nullity is 0.

Rank plus nullity

Recall the example map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ 2x+2y \end{pmatrix}$$

whose range space $\mathscr{R}(h)$ is the line $y = 2x$ and whose domain is organized into lines, $\mathscr{N}(h)$ is the line $y = -x$. There, an entire line's worth of domain vectors collapses to the single range point.



In moving from domain to range, this maps drops a dimension. We can account for it by thinking that each output point absorbs a one-dimensional set.

2.14 *Theorem* A linear map's rank plus its nullity equals the dimension of its domain.

The book contains the proof.

Example Consider this map $h\colon \mathbb{R}^3 \to \mathbb{R}$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{h}{\longmapsto} x/2 + y/5 + z$$

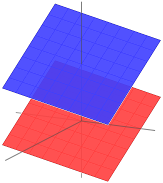
The null space is this plane.

$$\mathscr{N}(h) = h^{-1}(0) = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x/2 + y/5 + z = 0 \}$$

Other inverse image sets are also planes.

$$h^{-1}(1) = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x/2 + y/5 + z = 1 \} = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid z = 1 - x/2 - y/5 \}$$

This shows the inverse images $h^{-1}(0)$ and $h^{-1}(1)$ lined up on the z axis.



So h breaks the domain into stacked planes — any two inverse images $h^{-1}(r_1)$ and $h^{-1}(r_2)$ are collections of domain vectors whose endpoints form a plane. The only difference between these 2-dimensional subsets is where they sit in the stack, shown here as where they intersect z axis.

That is, h partitions the 3-dimensional domain into 2-dimensional sets, leaving 1 dimension of freedom, which matches the dimension of the map's range.

Example Projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$$

takes a 3-dimensional domain to a 2-dimensional range. Its null space is the z-axis, so its nullity is 1.

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This example shows the idea of the proof particularly clearly. Take the basis $B_N = \langle \vec{e}_3 \rangle$ for the null space.

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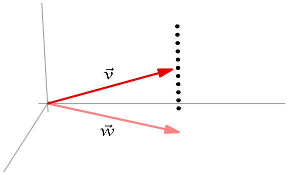
This example shows the idea of the proof particularly clearly. Take the basis $B_N = \langle \vec{e}_3 \rangle$ for the null space. Expand that to the basis $\mathcal{E}_3$ for the entire domain. On an input vector the action of π is

$$c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 \quad \mapsto \quad c_1\vec{e}_1 + c_2\vec{e}_2 + \vec{0}$$

and so the domain is organized by π into inverse images that are vertical lines, one-dimensional sets like the null space.



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$$ax^2 + bx + c \mapsto 2a \cdot x + b$$

has this range space

$$\mathscr{R}(d/dx) = \{d \cdot x + e \mid d, e \in \mathbb{R}\} = \mathcal{P}_1$$

(the linear polynomial $dx + e \in \mathcal{P}_1$ is the image of any antiderivative $(d/2)x^2 + ex + C$, where $C \in \mathbb{R}$). This is its null space.

$$\mathscr{N}(d/dx) = \{0x^2 + 0x + c \mid c \in \mathbb{R}\} = \{c \mid c \in \mathbb{R}\}$$

The rank is 2 while the nullity is 1, and they add to the domain's dimension 3.

Example The dilation function $d_3\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 3b \end{pmatrix}$$

has range space $\mathbb{R}^2$ and a trivial nullspace $\mathscr{N}(d_3) = \{\vec{0}\}$. So its rank is 2 and its nullity is 0.

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The book's next section is on computing linear maps, and we will compute more null spaces there.

2.18 *Lemma* Under a linear map, the image of a linearly dependent set is linearly dependent.

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Proof Suppose that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}_V$ with some c_i nonzero. Apply h to both sides: $h(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n)$ and $h(\vec{0}_V) = \vec{0}_W$. Thus we have $c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n) = \vec{0}_W$ with some c_i nonzero. QED

2.18 *Lemma* Under a linear map, the image of a linearly dependent set is linearly dependent.

Proof Suppose that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}_V$ with some c_i nonzero. Apply h to both sides: $h(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1h(\vec{v}_1) + \cdots + c_nh(\vec{v}_n)$ and $h(\vec{0}_V) = \vec{0}_W$. Thus we have $c_1h(\vec{v}_1) + \cdots + c_nh(\vec{v}_n) = \vec{0}_W$ with some c_i nonzero. QED

Example The trace function $\mathrm{Tr}\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$$

is linear. This set of matrices is dependent.

$$S = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\}$$

The three matrices map to 1, 0, and 2 respectively. The set $\{1, 0, 2\} \subseteq \mathbb{R}$ is linearly dependent.

A one-to-one homomorphism is an isomorphism

2.20 *Theorem* Where V is an n-dimensional vector space, these are equivalent statements about a linear map $h\colon V \to W$.

(1) h is one-to-one

(2) h has an inverse from its range to its domain that is a linear map

(3) $\mathscr{N}(h) = \{\vec{0}\,\}$, that is, $\text{nullity}(h) = 0$

(4) $\text{rank}(h) = n$

(5) if $\langle\vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ is a basis for V then $\langle h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n)\rangle$ is a basis for $\mathscr{R}(h)$

The book has the proof.

Transformations of $\mathbb{R}^2$

Lines go to lines

In a real space $\mathbb{R}^n$ a line through the origin is a set $\{r\cdot\vec{v}\mid r\in\mathbb{R}\}$ of multiples of a nonzero vector.

Consider a transformation $t\colon\mathbb{R}^n\to\mathbb{R}^n$. It is linear and so t's action

$$r\cdot\vec{v}\stackrel{t}{\longmapsto}r\cdot t(\vec{v})$$

sends members of the line $\{r\cdot\vec{v}\mid r\in\mathbb{R}\}$ in the domain to members of the line $\{s\cdot t(\vec{v})\mid s\in\mathbb{R}\}$ in the codomain.

Thus, under a transformation, lines through the origin map to lines through the origin. Further, the action of t is determined by its effect $t(\vec{v})$ on any nonzero element of the domain line.

Example Consider the line $y = 2x$ in the plane

$$\{r \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid r \in \mathbb{R}\}$$

and this transformation.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + 3y \\ 2x + 4y \end{pmatrix}$$

The map's effect on any vector in the line is easy to compute.

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

The linear map property $t(r \cdot \vec{v}) = r \cdot t(\vec{v})$ imposes a uniformity on t's action: t has twice the effect on $2\vec{v}$, three times the effect on $3\vec{v}$, etc.

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} 14 \\ 20 \end{pmatrix} \qquad \begin{pmatrix} -3 \\ -6 \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} -21 \\ -30 \end{pmatrix} \qquad \begin{pmatrix} r \\ 2r \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} 7r \\ 10r \end{pmatrix}$$

In short: the action of t on any nonzero $\vec{v}$ determines its action on any other vector $r\vec{v}$ in the line $[\{\vec{v}\}]$.

Pick one, any one

Every plane vector is in some line through the origin so to understand what $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ does to plane elements it suffices to understand what it does to lines through the origin.

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Pick one, any one

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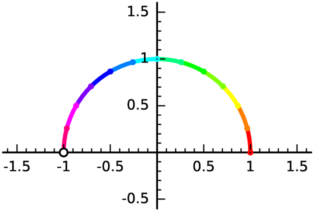
So one way to understand a transformation's action is to take a set containing one nonzero vector from each line through the origin, and describe where the transformation maps the elements of that set.

A natural set with one nonzero element from each line through the origin is the upper half unit circle (we will explain the colors below).

$$\{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \mid 0 \leqslant t < \pi\}$$

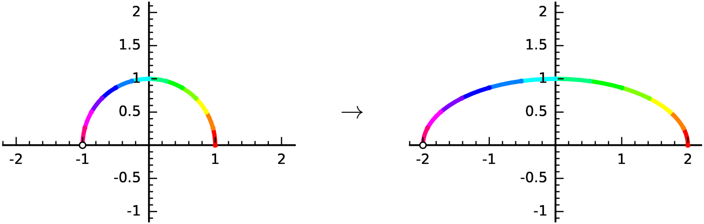


Dilate x

Example The map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x \\ y \end{pmatrix}$$

doubles the first coordinate while keeping the second coordinate constant. This shows the transformation of the upper half circle.

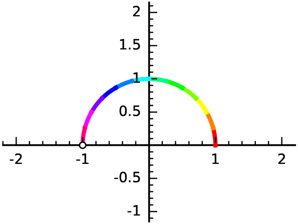


Reverse orientation

Example Here we dilate by a negative.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x \\ y \end{pmatrix}$$

The transformation of the upper half circle shows why we used the colors. In the domain they are, taken counterclockwise, red, orange, yellow, green, blue, indigo, violet. In the codomain, again taken counterclockwise, they do the opposite.



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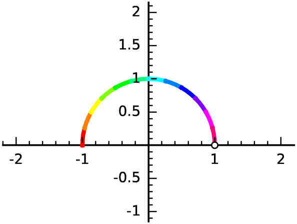
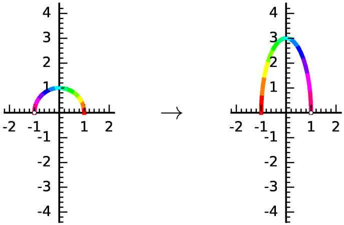


Combine dilations

Example Here we dilate both x and y.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x \\ 3y \end{pmatrix}$$

Again the color order reverses, in addition to the stretching along the y axis.

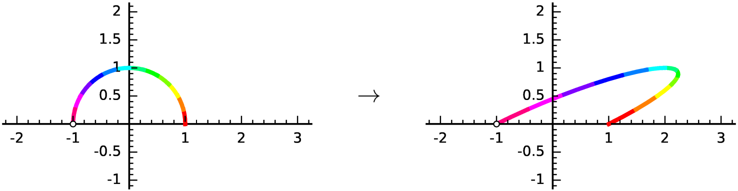


The two dilations combine independently in that the first coordinate of the output uses only x and the second coordinate of the output uses only y.

Skew

Example Next is a map with an output coordinate affected by both x and y.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + 2y \\ y \end{pmatrix}$$

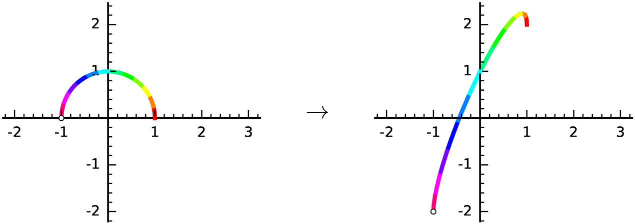


On the x axis, where $y = 0$, the output's first coordinate is the same as the input's first coordinate. However as we move away from the x axis the y's get larger, and the first coordinate of the output is increasingly affected. (One definition of *skew* is: having an oblique direction or position; slanting.)

Skew the other way

Example We can flip which output coordinate is affected by both x and y.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 2x + y \end{pmatrix}$$

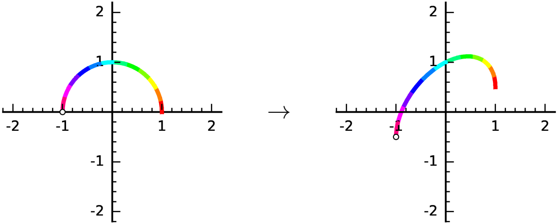


In addition to dilation we see clear rotation, for instance of the red input vector.

Same idea but with a smaller effect

Example

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ (1/2)x + y \end{pmatrix}$$



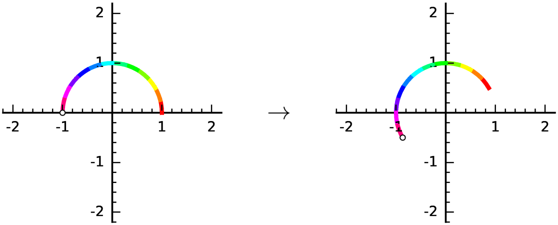
Observe that the rotation is not even. A red vector is rotated quite a bit but a green vector, near the y axis, is not rotated much. And right on the y-axis the vector is not rotated at all.

Pure roation

Example This rotates every vector counterclockwise through the angle θ.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos(\theta) \cdot x - \sin(\theta) \cdot y \\ \cos(\theta) \cdot x + \sin(\theta) \cdot y \end{pmatrix}$$

In this picture $\theta = \pi/6$.

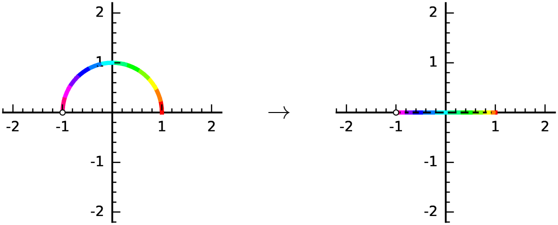


Projection

Example Maps can lose a dimension.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$$

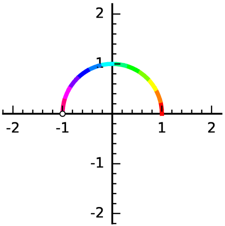
The output is one-dimensional.



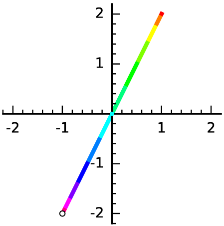
Example The map may project the input vector to a line that is not an axis.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 2x \end{pmatrix}$$

The two-dimensional input is sent to an output that is one-dimensional.



$\longrightarrow$

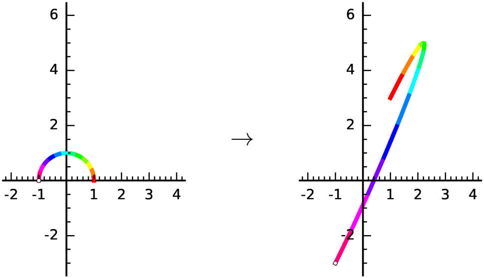


A generic map

Example An arbitrary map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix}$$

may have an action that is a mixture of the effects shown above.



This shows dilation, rotation, and orientation reversal.