Two.III Basis and Dimension

Linear Algebra, edition four

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Basis

Definition of basis

1.1 *Definition* A ***basis*** for a vector space is a sequence of vectors that is linearly independent and that spans the space.

Because a basis is a sequence, meaning that bases are different if they contain the same elements but in different orders, we denote it with angle brackets $\langle \vec{\beta}_1, \vec{\beta}_2, \ldots \rangle$.

Example This is a basis for $\mathbb{R}^2$.

$$\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$$

It is linearly independent.

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{aligned} c_1 + c_2 &= 0 \\ -c_1 + c_2 &= 0 \end{aligned} \implies c_1 = 0,\ c_2 = 0$$

And it spans $\mathbb{R}^2$ since

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \implies \begin{aligned} c_1 + c_2 &= x \\ -c_1 + c_2 &= y \end{aligned}$$

has the solution $c_1 = (1/2)x - (1/2)y$ and $c_2 = (1/2)x + (1/2)y$.

Example In the vector space of linear polynomials $\mathcal{P}_1 = \{a + bx \mid a, b \in \mathbb{R}\}$ one basis is $B = \langle 1 + x, 1 - x \rangle$.

Check that is a basis by verifying that it is linearly independent

$$0 = c_1(1 + x) + c_2(1 - x) \implies 0 = c_1 + c_2,\ 0 = c_1 - c_2 \implies c_1 = c_2 = 0$$

and that it spans the space.

$$a + bx = c_1(1 + x) + c_2(1 - x) \implies c_1 = (a + b)/2,\ c_2 = (a - b)/2$$

Example This is a basis for $\mathcal{M}_{2\times 2}$.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \rangle$$

This is another one.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \rangle$$

Example This is a basis for $\mathbb{R}^3$.

$$\mathcal{E}_3 = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

Calculus books sometimes call those $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$.

1.5 *Definition* For any $\mathbb{R}^n$

$$\mathcal{E}_n = \langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \rangle$$

is the *standard* (or *natural*) basis. We denote these vectors $\vec{e}_1, \ldots, \vec{e}_n$.

Checking that $\mathcal{E}_n$ is a basis for $\mathbb{R}^n$ is routine.

Although a basis is a sequence we will follow the common practice and refer to it as a set.

1.12 *Theorem* In any vector space, a subset is a basis if and only if each vector in the space can be expressed as a linear combination of elements of the subset in one and only one way.

Proof A sequence is a basis if and only if its vectors form a set that spans and that is linearly independent. A subset is a spanning set if and only if each vector in the space is a linear combination of elements of that subset in at least one way. Thus we need only show that a spanning subset is linearly independent if and only if every vector in the space is a linear combination of elements from the subset in at most one way.

Consider two expressions of a vector as a linear combination of the members of the subset. Rearrange the two sums, and if necessary add some $0 \cdot \vec{\beta}_i$ terms, so that the two sums combine the same $\vec{\beta}$'s in the same order: $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$ and $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + \cdots + d_n\vec{\beta}_n$. Now

$$c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + \cdots + d_n\vec{\beta}_n$$

holds if and only if

$$(c_1 - d_1)\vec{\beta}_1 + \cdots + (c_n - d_n)\vec{\beta}_n = \vec{0}$$

holds. So, asserting that each coefficient in the lower equation is zero is the same thing as asserting that $c_i = d_i$ for each i, that is, that every vector is expressible as a linear combination of the $\vec{\beta}$'s in a unique way. QED

Example This is a vector and basis for the vector space $\mathbb{R}^3$.

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3 \qquad B = \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle = \langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \rangle$$

Find how to express $\vec{v}$ as $c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + c_3\vec{\beta}_3$ by solving this system.

$$\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ c_1 + c_2 \phantom{{}+c_3} &= 2 \\ c_1 \phantom{{}+c_2+c_3} &= 3 \end{aligned}$$

By eye we see just what the Theorem says we will see: there is one and only one solution $c_1 = 3$, $c_2 = -1$, and $c_3 = -1$.

1.13 *Definition* In a vector space with basis B the *representation of $\vec{v}$ with respect to B* is the column vector of the coefficients used to express $\vec{v}$ as a linear combination of the basis vectors:

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_B$$

where $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ and $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$. The c's are the *coordinates of $\vec{v}$ with respect to B*.

The prior slide shows that where

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad B = \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle$$

we have this.

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}_B$$

Example Above we saw that in $\mathcal{P}_1 = \{a + bx \mid a, b \in \mathbb{R}\}$ one basis is $B = \langle 1 + x, 1 - x \rangle$. As part of that we computed the coefficients needed to express a member of $\mathcal{P}_1$ as a combination of basis vectors.

$$a + bx = c_1(1 + x) + c_2(1 - x) \implies c_1 = (a + b)/2,\ c_2 = (a - b)/2$$

For instance, the polynomial $3 + 4x$ has this expression

$$3 + 4x = (7/2) \cdot (1 + x) + (-1/2) \cdot (1 - x)$$

so its representation is this.

$$\mathrm{Rep}_B(3 + 4x) = \begin{pmatrix} 7/2 \\ -1/2 \end{pmatrix}$$

Example With respect to $\mathbb{R}^3$'s standard basis $\mathcal{E}_3$ the vector

$$\vec{v}=\begin{pmatrix}2\\-3\\1/2\end{pmatrix}$$

has this representation.

$$\mathrm{Rep}_{\mathcal{E}_3}(\vec{v})=\begin{pmatrix}2\\-3\\1/2\end{pmatrix}$$

In general, any $\vec{w}\in\mathbb{R}^n$ has $\mathrm{Rep}_{\mathcal{E}_n}(\vec{w})=\vec{w}$.

Dimension

Definition of dimension

2.1 *Definition* A vector space is *finite-dimensional* if it has a basis with only finitely many vectors.

Example The space $\mathbb{R}^3$ is finite-dimensional since it has a basis with three elements $\mathcal{E}_3$.

Example The space of quadratic polynomials $\mathcal{P}_2$ has at least one basis with finitely many elements, $\langle 1, x, x^2 \rangle$, so it is finite-dimensional.

Example The space $\mathcal{M}_{2\times 2}$ of 2×2 matrices is finite-dimensional. Here is one basis with finitely many members.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle$$

Note From this point on we will restrict our attention to vector spaces that are finite-dimensional. All the later examples, definitions, and theorems assume this of the spaces.

We will show that for any finite-dimensional space, all of its bases have the same number of elements.

Example Each of these is a basis for $\mathcal{P}_2$.

$$\begin{aligned}
B_0 &= \langle 1, 1+x, 1+x+x^2 \rangle \\
B_1 &= \langle 1+x+x^2, 1+x, 1 \rangle \\
B_2 &= \langle x^2, 1+x, 1-x \rangle \\
B_3 &= \langle 1, x, x^2 \rangle
\end{aligned}$$

Each has three elements.

Example Here are two different bases for $\mathcal{M}_{2\times 2}$.

$$\begin{aligned}
B_0 &= \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle \\
B_1 &= \langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rangle
\end{aligned}$$

Both have four elements.

All of a space's bases are the same size

2.4 *Theorem* In any finite-dimensional vector space, all bases have the same number of elements.

Note: the proof in the book is different. This one relies more on computation with coordinates.

Proof Fix a vector space with at least one finite basis. From among all of this space's bases, choose one $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ that has minimal size. We will show that any other basis $D = \langle \vec{\delta}_1, \vec{\delta}_2, \ldots \rangle$ also has n members. Because B has minimal size, D cannot have fewer than n vectors. We will argue that it cannot have more.

So suppose that $\vec{\delta}_1, \ldots, \vec{\delta}_{n+1}$ are distinct. The assumption that D is linearly independent gives that the only relationship $a_1\vec{\delta} + 1 + \cdots + a_{n+1}\vec{\delta}_{n+1} = \vec{0}$ is the one where each a_i is zero. We shall get a contradiction by finding that there is a nontrivial relationships among these $n+1$ vectors. First represent them with respect to B.

$$\mathrm{Rep}_B(\vec{\delta}_1) = \begin{pmatrix} c_{1,1} \\ \vdots \\ c_{n,1} \end{pmatrix} \quad \cdots \quad \mathrm{Rep}_B(\vec{\delta}_{n+1}) = \begin{pmatrix} c_{1,n+1} \\ \vdots \\ c_{n,n+1} \end{pmatrix}$$

Lemma 1.18 says that a relationship holds among the vectors $a_1\vec{\delta}_1 + \cdots + a_{n+1}\vec{\delta}_{n+1} = \vec{0}$ if and only if the same relationship holds among the representations.

$$a_1 \begin{pmatrix} c_{1,1} \\ \vdots \\ c_{n,1} \end{pmatrix} + \cdots + a_{n+1} \begin{pmatrix} c_{1,n+1} \\ \vdots \\ c_{n,n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Here the $c_{i,j}$ are fixed, since they are the coefficients in the representations of the given vectors, while we are looking for the a_i. Thus the above is a homogeneous linear system

$$\begin{aligned} c_{1,1}a_1 + \cdots + c_{1,n+1}a_{n+1} &= 0 \\ &\vdots \\ c_{n,1}a_1 + \cdots + c_{n,n+1}a_{n+1} &= 0 \end{aligned}$$

with more unknowns, $n+1$, than equations. Such a system does not have only one solution, it has infinitely many solutions. QED

Definition of dimension

2.5 *Definition* The *dimension* of a vector space is the number of vectors in any of its bases.

Example The vector space $\mathbb{R}^n$ has dimension n because that is how many members are in $\mathcal{E}_n$.

Example The vector space $\mathcal{P}_2$ has dimension 3 because one of its bases is $\langle 1, x, x^2 \rangle$. More generally, $\mathcal{P}_n$ has dimension $n+1$.

Example The vector space $\mathcal{M}_{n\times m}$ has dimension $n \cdot m$. A natural basis consists of matrices with a single 1 and the other entries 0's.

Example The solution set S of this system

$$\begin{aligned} x - y + z \phantom{{}+2w} &= 0 \\ -x + 2y - z + 2w &= 0 \\ -x + 3y - z + 4w &= 0 \end{aligned}$$

is a vector space (this is easy to check for any homogeneous system). Solving the system

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 2 & 0 \\ 1 & 3 & -1 & 4 & 0 \end{array}\right) \xrightarrow[\rho_1+\rho_3]{\rho_1+\rho_2} \xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

and parametrizing gives a basis of two vectors.

$$\{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot z + \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \cdot w \mid z, w \in \mathbb{R}\} \qquad B = \langle \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \rangle$$

So S is a vector space of dimension two.

2.10 *Corollary* No linearly independent set can have a size greater than the dimension of the enclosing space.

Proof The proof of Theorem 2.4 never uses that D spans the space, only that it is linearly independent. QED

Remark This is an example of a result that assumes the vector spaces are finite-dimensional without specifically saying so.

2.12 *Corollary* Any linearly independent set can be expanded to make a basis.

Proof If a linearly independent set is not already a basis then it must not span the space. Adding to the set a vector that is not in the span will preserve linear independence by Lemma II.1.15 . Keep adding until the resulting set does span the space, which the prior corollary shows will happen after only a finite number of steps. QED

2.13 *Corollary* Any spanning set can be shrunk to a basis.

Proof Call the spanning set S. If S is empty then it is already a basis (the space must be a trivial space). If $S = \{\vec{0}\}$ then it can be shrunk to the empty basis, thereby making it linearly independent, without changing its span.

Otherwise, S contains a vector $\vec{s}_1$ with $\vec{s}_1 \neq \vec{0}$ and we can form a basis $B_1 = \langle \vec{s}_1 \rangle$. If $[B_1] = [S]$ then we are done. If not then there is a $\vec{s}_2 \in [S]$ such that $\vec{s}_2 \notin [B_1]$. Let $B_2 = \langle \vec{s}_1, \vec{s_2} \rangle$; by Lemma II.1.15 this is linearly independent so if $[B_2] = [S]$ then we are done.

We can repeat this process until the spans are equal, which must happen in at most finitely many steps. QED

2.14 *Corollary* In an n-dimensional space, a set composed of n vectors is linearly independent if and only if it spans the space.

Proof First we will show that a subset with n vectors is linearly independent if and only if it is a basis. The 'if' is trivially true — bases are linearly independent. 'Only if' holds because a linearly independent set can be expanded to a basis, but a basis has n elements, so this expansion is actually the set that we began with.

To finish, we will show that any subset with n vectors spans the space if and only if it is a basis. Again, 'if' is trivial. 'Only if' holds because any spanning set can be shrunk to a basis, but a basis has n elements and so this shrunken set is just the one we started with. QED

Example This clearly spans the space.

$$\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle \subseteq \mathbb{R}^3$$

Because it has same number of elements as the dimension of the space, it is therefore a basis.

Vector Spaces and Linear Systems

Row space

3.1 *Definition* The *row space* of a matrix is the span of the set of its rows. The *row rank* is the dimension of this space, the number of linearly independent rows.

3.3 *Lemma* If two matrices A and B are related by a row operation

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} B \quad \text{or} \quad A \xrightarrow{k\rho_i} B \quad \text{or} \quad A \xrightarrow{k\rho_i + \rho_j} B$$

(for $i \neq j$ and $k \neq 0$) then their row spaces are equal. Hence, row-equivalent matrices have the same row space and therefore the same row rank.

Proof Corollary One.III.2.4 shows that when $A \longrightarrow B$ then each row of B is a linear combination of the rows of A. That is, in the above terminology, each row of B is an element of the row space of A. Then $\text{Rowspace}(B) \subseteq \text{Rowspace}(A)$ follows because a member of the set $\text{Rowspace}(B)$ is a linear combination of the rows of B, so it is a combination of combinations of the rows of A, and by the Linear Combination Lemma is also a member of $\text{Rowspace}(A)$.

For the other set containment, recall Lemma One.III.1.5 , that row operations are reversible so $A \longrightarrow B$ if and only if $B \longrightarrow A$. Then $\text{Rowspace}(A) \subseteq \text{Rowspace}(B)$ follows as in the previous paragraph. QED

3.3 *Lemma* The nonzero rows of an echelon form matrix make up a linearly independent set.

Proof Lemma One.III.2.5 says that no nonzero row of an echelon form matrix is a linear combination of the other rows. This result restates that using this chapter's terminology. QED

Example The matrix before Gauss's Method and the matrix after have equal row spaces.

$$M = \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \\ -1 & -2 & 2 & 2 & 0 \\ 2 & 4 & 5 & 2 & 9 \end{pmatrix} \xrightarrow[-2\rho_1+\rho_3]{\rho_1+\rho_2} \xrightarrow{-\rho_2+\rho_3} \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \\ 0 & 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The nonzero rows of the latter matrix form a basis for $\text{Rowspace}(M)$.

$$B = \langle \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 3 & 2 & 3 \end{pmatrix} \rangle$$

The row rank is 2.

So Gauss's Method produces a basis for the row space of a matrix. It has found the "repeat" information, that M's third row is three times the first plus the second, and eliminated that extra row.

Column space

3.6 *Definition* The *column space* of a matrix is the span of the set of its columns. The *column rank* is the dimension of the column space, the number of linearly independent columns.

Example This system

$$\begin{aligned} 2x + \quad 3y &= d_1 \\ -x + (1/2)y &= d_2 \end{aligned}$$

has a solution for those $d_1, d_2 \in \mathbb{R}$ that we can find to satisfy this vector equation.

$$x \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} + y \cdot \begin{pmatrix} 3 \\ 1/2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \qquad x, y \in \mathbb{R}$$

That is, the system has a solution if and only if the vector on the right is in the column space of this matrix.

$$\begin{pmatrix} 2 & 3 \\ -1 & 1/2 \end{pmatrix}$$

Transpose

3.8 *Definition* The *transpose* of a matrix is the result of interchanging its rows and columns, so that column j of the matrix A is row j of A^{T} and vice versa.

Example To find a basis for the column space of a matrix,

$$\begin{pmatrix} 2 & 3 \\ -1 & 1/2 \end{pmatrix}$$

transpose,

$$\begin{pmatrix} 2 & 3 \\ -1 & 1/2 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} 2 & -1 \\ 3 & 1/2 \end{pmatrix}$$

reduce,

$$\begin{pmatrix} 2 & -1 \\ 3 & 1/2 \end{pmatrix} \xrightarrow{(-3/2)\rho_1+\rho_2} \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$$

and transpose back.

$$\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$$

This basis

$$B = \langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle$$

shows that the column space is the entire vector space $\mathbb{R}^2$.

3.10 *Lemma* Row operations do not change the column rank.

The reason is that row operations do not change linear relationships between the columns. So row operations do not change the number of linearly unrelated columns.

The book has the full proof; here is an example. In this matrix, the second column minus the first is equal to the third.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 2 & 6 & 4 \end{pmatrix}$$

As we perform row operations that relationship between the columns continues to hold.

$$\xrightarrow{-2\rho_1+\rho_3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & 2 & 2 \end{pmatrix} \xrightarrow{-(2/3)\rho_2+\rho_3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

3.11 *Theorem* For any matrix, the row rank and column rank are equal.

Proof Bring the matrix to reduced echelon form. Then the row rank equals the number of leading entries since that equals the number of nonzero rows. Then also, the number of leading entries equals the column rank because the set of columns containing leading entries consists of some of the $\vec{e}_i$'s from a standard basis, and that set is linearly independent and spans the set of columns. Hence, in the reduced echelon form matrix, the row rank equals the column rank, because each equals the number of leading entries.

But Lemma 3.3 and Lemma 3.10 show that the row rank and column rank are not changed by using row operations to get to reduced echelon form. Thus the row rank and the column rank of the original matrix are also equal. QED

3.12 *Definition* The *rank* of a matrix is its row rank or column rank.

Example The column rank of this matrix

$$\begin{pmatrix} 2 & -1 & 3 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 4 & -1 \\ 4 & -2 & 6 & 2 & 0 & 2 \\ 1 & 0 & 3 & 0 & 0 & 2 \end{pmatrix}$$

is 3. Its largest set of linearly independent columns is size 3 because that's the size of its largest set of linearly independent rows.

$$\xrightarrow[\substack{-2\rho_1+\rho_3 \\ -(1/2)\rho_1+\rho_4}]{-(3/2)\rho_1+\rho_2} \xrightarrow{-(1/3)\rho_2+\rho_4} \xrightarrow{\rho_3\leftrightarrow\rho_4} \begin{pmatrix} 2 & -1 & 3 & 1 & 0 & 1 \\ 0 & 3/2 & -7/2 & -1/2 & 4 & -5/2 \\ 0 & 0 & 8/3 & -1/3 & -4/3 & 7/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3.13 *Theorem* For linear systems with n unknowns and with matrix of coefficients A, the statements

(1) the rank of A is r

(2) the vector space of solutions of the associated homogeneous system has dimension $n - r$

are equivalent.

Proof The rank of A is r if and only if Gaussian reduction on A ends with r nonzero rows. That's true if and only if echelon form matrices row equivalent to A have r-many leading variables. That in turn holds if and only if there are $n - r$ free variables. QED

3.14 *Corollary* Where the matrix A is $n \times n$, these statements

(1) the rank of A is n
(2) A is nonsingular
(3) the rows of A form a linearly independent set
(4) the columns of A form a linearly independent set
(5) any linear system whose matrix of coefficients is A has one and only one solution

are equivalent.

Proof Clearly (1) $\iff$ (2) $\iff$ (3) $\iff$ (4). The last, (4) $\iff$ (5), holds because a set of n column vectors is linearly independent if and only if it is a basis for $\mathbb{R}^n$, but the system

$$c_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}$$

has a unique solution for all choices of $d_1, \ldots, d_n \in \mathbb{R}$ if and only if the vectors of a's on the left form a basis. QED