Three.I Isomorphisms

Linear Algebra, edition four

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Definition

Example We have the intuition that the vector spaces $\mathbb{R}^2$ and $\mathcal{P}_1$ are "the same," in that they are two-component spaces. For instance

$$\begin{pmatrix}1\\2\end{pmatrix} \quad \text{is just like} \quad 1+2x,$$

$$\text{and} \quad \begin{pmatrix}-3\\1/2\end{pmatrix} \quad \text{is just like} \quad -3+(1/2)x,$$

etc. What makes the spaces alike, not just the sets, is that the association persists through the operations: this illustrates addition

$$\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}-3\\1/2\end{pmatrix}=\begin{pmatrix}-2\\5/2\end{pmatrix}$$

$$\text{is just like} \quad (1+2x)+(-3+(1/2)x)=-2+(5/2)x$$

and this illustrates scalar multiplication.

$$3\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3\\6\end{pmatrix} \quad \text{is just like} \quad 3(1+2x)=3+6x$$

Example Similarly, we can link each two-tall vector with a linear polynomial.

$$\begin{pmatrix} a \\ b \end{pmatrix} \quad \longleftrightarrow \quad a + bx$$

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This association holds through the vector space operations of addition

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix}$$
$$\longleftrightarrow \quad (a_1 + b_1 x) + (a_2 + b_2 x) = (a_1 + a_2) + (b_1 + b_2)x$$

and scalar multiplication.

$$r \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ra \\ rb \end{pmatrix} \quad \longleftrightarrow \quad r(a + bx) = (ra) + (rb)x$$

We say that the association *preserves the structure* of the spaces.

Example We can think of $\mathcal{M}_{2\times 2}$ as "the same" as $\mathbb{R}^4$ if we associate in this way.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

For instance, these are corresponding elements.

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$$

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This association persists under addition.

$$\begin{aligned} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} &= \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} \\ &\longleftrightarrow \quad \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{pmatrix} \end{aligned}$$

Here is an example of addition being preserved under this association.

$$\begin{pmatrix}1 & -1\\ 2 & -2\end{pmatrix} + \begin{pmatrix}0 & 4\\ 3 & -3\end{pmatrix} = \begin{pmatrix}1 & 3\\ 5 & -5\end{pmatrix}$$

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The association also persists through scalar multiplication.

$$r \cdot \begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}ra & rb\\ rc & rd\end{pmatrix} \quad \longleftrightarrow \quad r \cdot \begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix}ra\\ rb\\ rc\\ rd\end{pmatrix}$$

This illustrates.

$$2 \cdot \begin{pmatrix}1 & -1\\ 2 & -2\end{pmatrix} = \begin{pmatrix}2 & -2\\ 4 & -4\end{pmatrix} \quad \longleftrightarrow \quad 2 \cdot \begin{pmatrix}1\\ -1\\ 2\\ -1\end{pmatrix} = \begin{pmatrix}2\\ -2\\ 4\\ -4\end{pmatrix}$$

Isomorphism

1.3 *Definition* An *isomorphism* between two vector spaces V and W is a map $f\colon V \to W$ that

1) is a correspondence: f is one-to-one and onto;
2) *preserves structure:* if $\vec{v}_1, \vec{v}_2 \in V$ then

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$

and if $\vec{v} \in V$ and $r \in \mathbb{R}$ then

$$f(r\vec{v}) = rf(\vec{v})$$

(we write $V \cong W$, read "V is isomorphic to W", when such a map exists).

How-to

To verify that $f\colon V\to W$ is an isomorphism, do these four.

- To show that f is one-to-one, assume that $\vec{v}_1,\vec{v}_2\in V$ are such that $f(\vec{v}_1)=f(\vec{v}_2)$ and derive that $\vec{v}_1=\vec{v}_2$.
- To show that f is onto, let $\vec{w}$ be an element of W and find a $\vec{v}\in V$ such that $f(\vec{v})=\vec{w}$.
- To show that f preserves addition, check that for all $\vec{v}_1,\vec{v}_2\in V$ we have $f(\vec{v}_1+\vec{v}_2)=f(\vec{v}_1)+f(\vec{v}_2)$.
- To show that f preserves scalar multiplication, check that for all $\vec{v}\in V$ and $r\in\mathbb{R}$ we have $f(r\cdot\vec{v})=r\cdot f(\vec{v})$.

The first two cover condition (1), that the spaces correspond, that for each member of W there exactly one assocated member of V. The latter two cover (2), that the map preserves structure. For these two, the intuition is in the discussion above. (Later section cover these two at length.)

Example The space of quadratic polynomials $\mathcal{P}_2$ is isomorphic to $\mathbb{R}^3$ under this map.

$$f(a_0 + a_1x + a_2x^2) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

Here are two examples of the action of f.

$$f(1 + 2x + 3x^2) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad f(3 + 4x^2) = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

To verify that f is an isomorphism we must check condition (1), that f is a correspondence, and condition (2), that f preserves structure.

The first part of the definition's clause (1) is that f is one-to-one. We suppose $f(\vec{v}_1) = f(\vec{v}_2)$, that the function yields the same output on two inputs $f(a_0 + a_1x + a_2x^2) = f(b_0 + b_1x + b_2x^2)$. From that, we must derive that the two inputs are equal. The definition of f gives

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

and two column vectors are equal only if their entries are equal $a_0 = b_0$, $a_1 = b_1$, and $a_2 = b_2$. Thus the original inputs are equal $\vec{v}_1 = a_0 + a_1x + a_2x^2 = b_0 + b_1x + b_2x^2 = \vec{v}_2$. So f is one-to-one.

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The second part of (1) is that f is onto. We consider an element of the codomain

$$\vec{w} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^3$$

and produce an element of the domain that maps to it. Observe that $\vec{w}$ is the image under f of the member $\vec{v} = a_0 + a_1x + a_2x^2$ of the domain. Thus f is onto.

The definition's clause (2) also has two halves. First we show that f preserves addition. Consider f acting on the sum of two elements of the domain.

$$\begin{aligned}f(\,(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)\,)\\ = f(\,(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2\,)\end{aligned}$$

The definition of f gives

$$= \begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$

and that equals

$$= \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

which gives

$$= f(\,a_0 + a_1x + a_2x^2) + f(b_0 + b_1x + b_2x^2\,)$$

as required.

We finish by checking that f preserves scalar multiplication. This is similar to the check for addition.

$$\begin{aligned}
f(r\cdot(a_0+a_1x+a_2x^2)) &= f(\,(ra_0)+(ra_1)x+(ra_2)x^2\,) \\
&= \begin{pmatrix} ra_0 \\ ra_1 \\ ra_2 \end{pmatrix} \\
&= r\cdot\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \\
&= r\cdot f(\,a_0+a_1x+a_2x^2\,)
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&= r\cdot\begin{pmatrix} a_0\\ a_1\\ a_2 \end{pmatrix}\\
&= r\cdot f(\,a_0+a_1x+a_2x^2\,)
\end{aligned}
$$

So the function f is an isomorphism. Because there is an isomorphism, the two spaces are isomorphic $\mathcal{P}_2 \cong \mathbb{R}^3$.

Example Consider these two vector spaces (under the natural operations)

$$V=\{\begin{pmatrix}a&b\\c&0\end{pmatrix}\mid a,b,c\in\mathbb{R}\}\qquad W=\{(x\;\;y\;\;z)\mid x,y,z\in\mathbb{R}\}$$

and consider this function.

$$\begin{pmatrix}a&b\\c&0\end{pmatrix}\stackrel{f}{\longmapsto}(b\;\;2a\;\;a+c)$$

Here is an example of the map's action.

$$\begin{pmatrix}3&2\\1&0\end{pmatrix}\stackrel{f}{\longmapsto}(2\;\;6\;\;4)$$

We will verify that f is an isomorphism.

To show that f is one-to-one, suppose that

$$f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix}) = f(\begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix})$$

Then $(b_1 \;\; 2a_1 \;\; a_1 + c_1) = (b_2 \;\; 2a_2 \;\; a_2 + c_2)$. The first entries give that $b_1 = b_2$, the second entries that $a_1 = a_2$, and with that the third entries give that $c_1 = c_2$.

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$$\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix}$$

To show that f is onto, consider a member of W.

$$\vec{w} = (x \;\; y \;\; z)$$

We must find a $\vec{v}$ so that $f(\vec{v}) = \vec{w}$. The map sends the upper right entry of the input to the first entry of the output, so the upper right of $\vec{v}$ is x. Similarly, the upper left of $\vec{v}$ is $(1/2)y$. With that, the lower left is $z - (1/2)y$.

$$(x \;\; y \;\; z) = f(\begin{pmatrix} y/2 & x \\ z - y/2 & 0 \end{pmatrix})$$

To show that f preserves addition, assume

$$f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix}) = f(\begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & 0 \end{pmatrix})$$

which equals $(b_1 + b_2 \;\; 2(a_1 + a_2) \;\; (a_1 + a_2) + (c_1 + c_2))$. In turn, that equals this.

$$(b_1 \;\; 2a_1 \;\; a_1 + c_1) + (b_2 \;\; 2a_2 \;\; a_2 + c_2) = f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix}) + f(\begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix})$$

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Preservation of scalar multiplication is similar.

$$\begin{aligned} f(r \cdot \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}) &= f(\begin{pmatrix} ra & rb \\ rc & 0 \end{pmatrix}) \\ &= (rb \;\; 2ra \;\; ra + rc) \\ &= r \cdot (b \;\; 2a \;\; a + c) \\ &= r \cdot f(\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}) \end{aligned}$$

Preservation is special

Many functions do not preserve addition and scalar multiplication. For instance, $f\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$$

does not preserve addition since the sum done one way

$$f(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}) = f(\begin{pmatrix} 3 \\ 0 \end{pmatrix}) = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$$

gives a different result than the sum done the other way.

$$f(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) + f(\begin{pmatrix} 2 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

Special case: Automorphisms

1.7 *Definition* An *automorphism* is an isomorphism of a space with itself.

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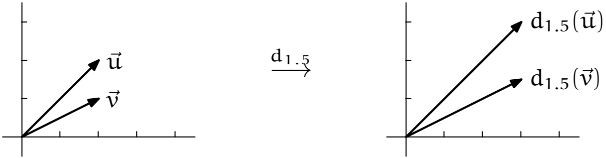

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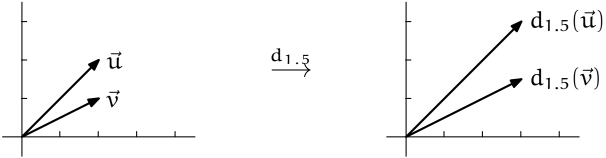
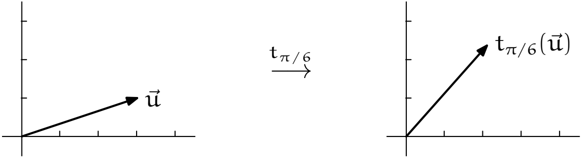
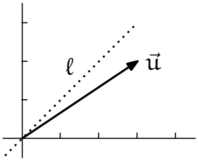


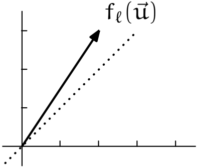
Another automorphism is a *rotation* or *turning map*, $t_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates all vectors through an angle θ.



A third type of automorphism of $\mathbb{R}^2$ is a map $f_\ell\colon \mathbb{R}^2 \to \mathbb{R}^2$ that *flips* or *reflects* all vectors over a line ℓ through the origin.



$\xrightarrow{f_\ell}$



Checking that each is an isomorphism is an exercise.

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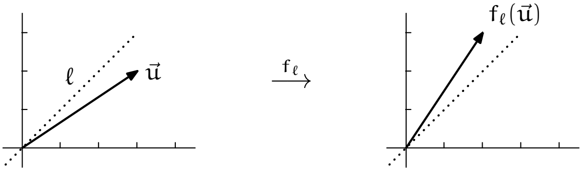

Checking that each is an isomorphism is an exercise.

Why study automorphisms? Isn't it trivial that the plane is just like itself?

Consider the family of automorphisms t_Θ rotating all vectors counterclockwise. They makes precise the intuition that the plane is uniform — that space near the x-axis is just like space near the y-axis.

So one lesson is that we can use maps to describe relationships between spaces. If the maps are isomorphisms then this relation makes precise the intuition "just like".

A second lesson is that while there is an obvious automorphism of $\mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

there are reasons to consider maps other than the obvious one.

1.10 *Lemma* An isomorphism maps a zero vector to a zero vector.

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Proof Where $f\colon V \to W$ is an isomorphism, fix some $\vec{v} \in V$. Then $f(\vec{0}_V) = f(0 \cdot \vec{v}) = 0 \cdot f(\vec{v}) = \vec{0}_W$. QED

1.11 *Lemma* For any map $f\colon V \to W$ between vector spaces these statements are equivalent.

(1) f preserves structure

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2) \quad \text{and} \quad f(c\vec{v}) = c\, f(\vec{v})$$

(2) f preserves linear combinations of two vectors

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2)$$

(3) f preserves linear combinations of any finite number of vectors

$$f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v}_n)$$

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Proof Since the implications (3) $\implies$ (2) and (2) $\implies$ (1) are clear, we need only show that (1) $\implies$ (3). So assume statement (1). We will prove (3) by induction on the number of summands n.

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The one-summand base case, that $f(c\vec{v}_1) = c\, f(\vec{v}_1)$, is covered by the second clause of statement (1).

For the inductive step assume that statement (3) holds whenever there are k or fewer summands. Consider the $k+1$-summand case. Use the first half of (1) to break the sum along the final '$+$'.

$$f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1}) = f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

Use the inductive hypothesis to break up the k-term sum on the left.

$$= f(c_1\vec{v}_1) + \cdots + f(c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

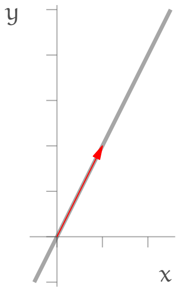
Now the second half of (1) gives

$$= c_1\, f(\vec{v}_1) + \cdots + c_k\, f(\vec{v}_k) + c_{k+1}\, f(\vec{v}_{k+1})$$

when applied $k+1$ times. QED

Example The line through the origin with slope two is a vector space.

$$L = \{ \begin{pmatrix} x \\ y \end{pmatrix} \mid y = 2x \} = \{ \begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R} \}$$



The parametrization

$$L = \{ \begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R} \} = \{ t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid t \in \mathbb{R} \}$$

suggests that L is just like $\mathbb{R}$: there is a point on L associated with $1 \in \mathbb{R}$, a point associated with $2 \in \mathbb{R}$, etc. We will show that this function is an isomorphism between L and $\mathbb{R}^1$.

$$f(\begin{pmatrix} t \\ 2t \end{pmatrix}) = t$$

To verify that f is one-to-one suppose that f maps two members of L to the same output.

$$f(\,t_1 \cdot \begin{pmatrix}1\\2\end{pmatrix}\,) = f(\,t_2 \cdot \begin{pmatrix}1\\2\end{pmatrix}\,)$$

By the definition of f we have $t_1 = t_2$ and so the two input members of L are equal.

To check that f is onto consider a member of the codomain, $r \in \mathbb{R}$. There is a member of the domain L that maps to it, namely this one.

$$f(\,r \cdot \begin{pmatrix}1\\2\end{pmatrix}\,) = r$$

To finish, we combine the two structure checks, using the lemma's (2).

$$\begin{aligned} f(\,c_1 \cdot \begin{pmatrix}t_1\\2t_1\end{pmatrix} + c_2 \cdot \begin{pmatrix}t_2\\2t_2\end{pmatrix}\,) &= f(\,(c_1t_1 + c_2t_2) \cdot \begin{pmatrix}1\\2\end{pmatrix}\,) \\ &= c_1t_1 + c_2t_2 = c_1 \cdot f(\,\begin{pmatrix}t_1\\2t_1\end{pmatrix}\,) + c_2 \cdot f(\,\begin{pmatrix}t_2\\2t_2\end{pmatrix}\,) \end{aligned}$$

Dimension characterizes isomorphism

2.1 *Lemma* The inverse of an isomorphism is also an isomorphism.

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Proof Suppose that V is isomorphic to W via $f\colon V \to W$. An isomorphism is a correspondence between the sets so f has an inverse function $f^{-1}\colon W \to V$ that is also a correspondence.

We will show that because f preserves linear combinations, so also does f^{-1}. Suppose that $\vec{w}_1, \vec{w}_2 \in W$. Because it is an isomorphism, f is onto and there are $\vec{v}_1, \vec{v}_2 \in V$ such that $\vec{w}_1 = f(\vec{v}_1)$ and $\vec{w}_2 = f(\vec{v}_2)$. Then

$$\begin{aligned}
f^{-1}(c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2) &= f^{-1}\big(\, c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\,\big) \\
&= f^{-1}\big(\, f\big(c_1\vec{v}_1 + c_2\vec{v}_2\big)\,\big) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1 \cdot f^{-1}(\vec{w}_1) + c_2 \cdot f^{-1}(\vec{w}_2)
\end{aligned}$$

since $f^{-1}(\vec{w}_1) = \vec{v}_1$ and $f^{-1}(\vec{w}_2) = \vec{v}_2$. With that, by Lemma 1.11 's second statement, this map preserves structure. QED

Example We saw earlier that this line through the origin subspace of $\mathbb{R}^2$

$$L = \{ \begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R} \}$$

is isomorphic to $\mathbb{R}^1$ via this function.

$$f(t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}) = t$$

The inverse $f^{-1} \colon \mathbb{R} \to L$ given by

$$f^{-1}(r) = r \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} r \\ 2r \end{pmatrix}$$

is also an isomorphism.

2.2 *Theorem* Isomorphism is an equivalence relation between vector spaces.

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Proof We must prove that the relation is symmetric, reflexive, and transitive.

To check reflexivity, that any space is isomorphic to itself, consider the identity map. It is clearly one-to-one and onto. This shows that it preserves linear combinations.

$$\mathrm{id}(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1 \cdot \mathrm{id}(\vec{v}_1) + c_2 \cdot \mathrm{id}(\vec{v}_2)$$

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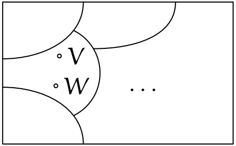
Symmetry, that if V is isomorphic to W then also W is isomorphic to V, holds by Lemma 2.1 since each isomorphism map from V to W is paired with an isomorphism from W to V.

To finish we must check transitivity, that if V is isomorphic to W and W is isomorphic to U then V is isomorphic to U. Let $f\colon V \to W$ and $g\colon W \to U$ be isomorphisms. Consider their composition $g \circ f\colon V \to U$. Because the composition of correspondences is a correspondence, we need only check that the composition preserves linear combinations.

$$\begin{aligned}
g \circ f\,\bigl(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2\bigr) &= g\bigl(\, f(\, c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 \,)\,\bigr) \\
&= g\bigl(\, c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\,\bigr) \\
&= c_1 \cdot g\bigl(f(\vec{v}_1)\bigr) + c_2 \cdot g\bigl(f(\vec{v}_2)\bigr) \\
&= c_1 \cdot (g \circ f)\,(\vec{v}_1) + c_2 \cdot (g \circ f)\,(\vec{v}_2)
\end{aligned}$$

Thus the composition is an isomorphism. QED

So the collection of spaces is partitioned into classes with two spaces in the same class if and only if they are isomorphic.



2.3 *Theorem* Vector spaces are isomorphic if and only if they have the same dimension.

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2.4 *Lemma* If spaces are isomorphic then they have the same dimension.

2.5 *Lemma* If spaces have the same dimension then they are isomorphic.

We prove them separately below.

2.4 *Lemma* If spaces are isomorphic then they have the same dimension.

Proof We shall show that an isomorphism of two spaces gives a correspondence between their bases. That is, we shall show that if $f\colon V \to W$ is an isomorphism and a basis for the domain V is $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ then its image $D = \langle f(\vec{\beta}_1), \dots, f(\vec{\beta}_n) \rangle$ is a basis for the codomain W. (The other half of the correspondence, that for any basis of W the inverse image is a basis for V, follows from the fact that f^{-1} is also an isomorphism and so we can apply the prior sentence to f^{-1}.)

To see that D spans W, fix any $\vec{w} \in W$. Because f is an isomorphism it is onto and so there is a $\vec{v} \in V$ with $\vec{w} = f(\vec{v})$. Expand $\vec{v}$ as a combination of basis vectors.

$$\vec{w} = f(\vec{v}) = f(v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n) = v_1 \cdot f(\vec{\beta}_1) + \cdots + v_n \cdot f(\vec{\beta}_n)$$

For linear independence of D, if

$$\vec{0}_W = c_1 f(\vec{\beta}_1) + \cdots + c_n f(\vec{\beta}_n) = f(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n)$$

then, since f is one-to-one and so the only vector sent to $\vec{0}_W$ is $\vec{0}_V$, we have that $\vec{0}_V = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$, which implies that all of the c's are zero.

QED

2.5 *Lemma* If spaces have the same dimension then they are isomorphic.

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Proof We will prove that any space of dimension n is isomorphic to $\mathbb{R}^n$. Then we will have that all such spaces are isomorphic to each other by transitivity, which was shown in Theorem 2.2 .

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Proof We will prove that any space of dimension n is isomorphic to $\mathbb{R}^n$. Then we will have that all such spaces are isomorphic to each other by transitivity, which was shown in Theorem 2.2 .

Let V be n-dimensional. Fix a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ for the domain V. Consider the operation of representing the members of V with respect to B as a function from V to $\mathbb{R}^n$.

$$\vec{v} = v_1 \vec{\beta}_1 + \cdots + v_n \vec{\beta}_n \overset{\text{Rep}_B}{\longmapsto} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

This function is one-to-one because if

$$\mathrm{Rep}_B(u_1\vec{\beta}_1 + \cdots + u_n\vec{\beta}_n) = \mathrm{Rep}_B(v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n)$$

then

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

and so $u_1 = v_1$, ..., $u_n = v_n$, implying that the original arguments $u_1\vec{\beta}_1 + \cdots + u_n\vec{\beta}_n$ and $v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n$ are equal.

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This function is onto; any member of $\mathbb{R}^n$

$$\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

is the image of some $\vec{v} \in V$, namely $\vec{w} = \mathrm{Rep}_B(w_1\vec{\beta}_1 + \cdots + w_n\vec{\beta}_n)$.

Finally, this function preserves structure.

$$\begin{aligned}
\mathrm{Rep}_B(r\cdot\vec{u}+s\cdot\vec{v}) &= \mathrm{Rep}_B(\,(ru_1+sv_1)\vec{\beta}_1+\cdots+(ru_n+sv_n)\vec{\beta}_n\,) \\
&= \begin{pmatrix} ru_1+sv_1 \\ \vdots \\ ru_n+sv_n \end{pmatrix} \\
&= r\cdot\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + s\cdot\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\
&= r\cdot\mathrm{Rep}_B(\vec{u})+s\cdot\mathrm{Rep}_B(\vec{v})
\end{aligned}$$

Therefore Rep_B is an isomorphism. Consequently any n-dimensional space is isomorphic to $\mathbb{R}^n$. QED

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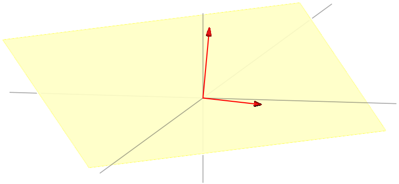
\end{aligned}$$

Therefore Rep_B is an isomorphism. Consequently any n-dimensional space is isomorphic to $\mathbb{R}^n$. QED

Note The second paragraph's representation map Rep_B is a well-defined function since for each basis, every vector $\vec{v}$ has a unique representation with respect to that basis.

Example The plane $2x - y + z = 0$ through the origin in $\mathbb{R}^3$ is a vector space (under the natural operations).



Describe the space as a span by taking that to be a one-equation linear system and parametrizing $x = (1/2)y - (1/2)z$.

$$P = \{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R} \}$$

Clearly the set with those two vectors is linearly independent so it is a basis.

$$B = \langle \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \rangle$$

Here is the basis from the prior slide.

$$B = \langle \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \rangle$$

Because its basis has two vectors, P is a dimension 2 space.

The second lemma's proof shows that this is an isomorphism: the map $f\colon P \to \mathbb{R}^2$ that associates each element $\vec{v} \in P$ with its representation $\mathrm{Rep}_B(\vec{v}) \in \mathbb{R}^2$.

To illustrate the association, here is an example of its action on an $\mathbb{R}^2$ vector picked at random.

$$\vec{v}_1 = \begin{pmatrix} 7/2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot 3 + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot (-4) \quad \stackrel{f}{\longmapsto} \quad \mathrm{Rep}_B(\vec{v}_1) = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

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Another (as with the prior one, $\vec{v}_2$ is picked at random from $\mathbb{R}^2$).

$$\vec{v}_2 = \begin{pmatrix} -17/4 \\ 1/2 \\ 9 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot (1/2) + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot 9 \quad \stackrel{f}{\longmapsto} \quad \mathrm{Rep}_B(\vec{v}_2) = \begin{pmatrix} 1/2 \\ 9 \end{pmatrix}$$

The first lemma's proof shows that any isomorphism takes bases to bases: starting with basis vectors $\vec{\beta}_i$ for the domain and applying an isomorphism f gives basis vectors $f(\vec{\beta}_i)$ for the range.

For the isomorphism from the prior slide we have

$$\vec{\beta}_1 = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot 1 + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot 0 \quad \stackrel{f}{\longmapsto} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\vec{\beta}_2 = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot 0 + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot 1 \quad \stackrel{f}{\longmapsto} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which together make the basis $\mathcal{E}_2$ for $\mathbb{R}^2$.

2.8 *Corollary* Each finite-dimensional vector space is isomorphic to one and only one of the $\mathbb{R}^n$.

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Thus the real spaces $\mathbb{R}^n$ form a set of canonical representatives of the isomorphism classes — every isomorphism class contains one and only one $\mathbb{R}^n$.