Four.II Geometry of Determinants

Linear Algebra

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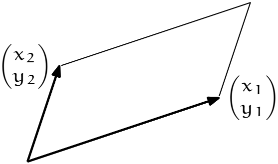
Determinants as size functions

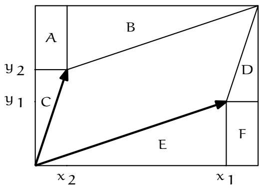
Box

This parallelogram is defined by the two vectors.



1.1 *Definition* In $\mathbb{R}^n$ the *box* (or *parallelepiped*) formed by $\langle \vec{v}_1, \ldots, \vec{v}_n \rangle$ is the set $\{t_1\vec{v}_1 + \cdots + t_n\vec{v}_n \mid t_1, \ldots, t_n \in [0 \ldots 1]\}$.

Area



$$
\begin{aligned}
\text{box area} &= \text{rectangle area} - \text{area of } A - \cdots - \text{area of } F \\
&= (x_1 + x_2)(y_1 + y_2) - x_2 y_1 - x_1 y_1/2 \\
&\quad - x_2 y_2/2 - x_2 y_2/2 - x_1 y_1/2 - x_2 y_1 \\
&= x_1 y_2 - x_2 y_1
\end{aligned}
$$

The determinant of this matrix gives the size of the box formed by the matrix's columns.

$$
\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1
$$

Determinant as a function of the columns

We now switch from considering the determinant as a function of the rows to considering it as a function of the columns.

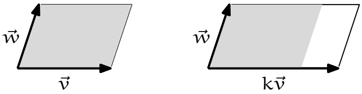
Because the determinant of the transpose equals the determinant of the matrix, the row operation conditions in the definition translate over to column operation conditions. That is, (1) a determinant is unchanged by a column combination: where A is square and

$$A \xrightarrow{k\mathrm{col}_i + \mathrm{col}_j} \hat{A}$$

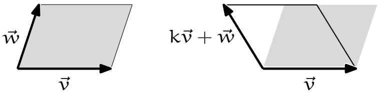
(with $i \neq j$) then $\det(A) = \det \hat{A}$, (2) a column swap changes the determinant's sign, and (3) multiplying a column by a scalar multiplies the entire determinant by that scalar. Condition (4), that the determinant of the identity matrix is 1, isn't about row or column operations so it still applies, without translation.

Geometric interpretation of the definition

The third condition on a determinant, stated in column form, is that rescaling a column rescales the entire determinant $\det(\ldots, k\vec{v}_i, \ldots) = k \cdot \det(\ldots, \vec{v}_i, \ldots)$. This fits our program of arguing that determinant conditions make good axioms for a function measuring the size of a box: if we scale a column by a factor k then the size of the box scales by that factor.



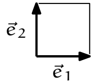
The determinant's first condition is that it is unaffected by combining columns. The picture



shows that the box formed by the two vectors $\vec{v}$ and $k\vec{v} + \vec{w}$ is slanted at a different angle than the one formed by $\vec{v}$ and $\vec{w}$ but the two boxes have the same base and height, and hence the same area.

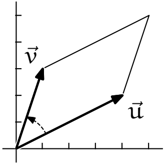
As we noted after the determinant's definition, its second condition is a consequence of the others so we leave it aside for a moment.

The final condition is that the determinant of the identity matrix is 1. This also fits with our program of arguing that the conditions in the definition of determinant are good ones for the function giving the size of the box formed by the columns of the matrix.

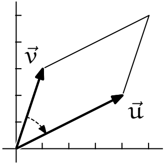


Orientation

1.2 *Remark* The second condition in the definition is that swapping changes the sign of the determinant. Consider these pictures, using the same pair of vectors.



$$\begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = 10$$



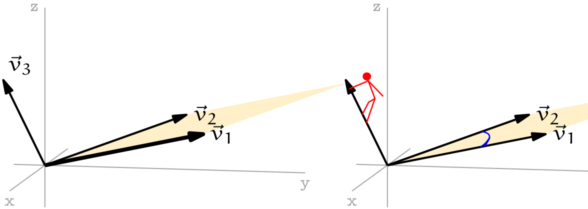
$$\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = -10$$

On the left, $\vec{u}$ is the first column in the matrix and we get a positive size. On the right, $\vec{v}$ is first and we get a negative size. On the left, the arc from first vector to second is counterclockwise while on the right, the arc from first to second is clockwise. The sign returned by the determinant reflects the *orientation* or *sense* of the box.

More on orientation: $\mathbb{R}^3$

These two vectors span a plane. The plane divides three-space into two parts, the part above and the part below.

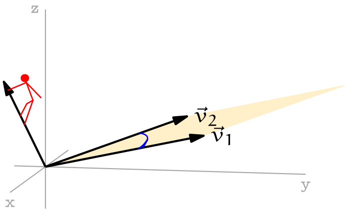
$$\vec{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$



Take a $\vec{v}_3$ above that plane. More precisely, the $\vec{v}_3$ shown is on the side of the plane with the property that a person at the tip of $\vec{v}_3$ looking at the arc from $\vec{v}_1$ to $\vec{v}_2$ sees that arc as counterclockwise. Any such vector defines a box that is positive-sized.

$$\vec{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \qquad \begin{vmatrix} 1 & -2 & 0 \\ 4 & 3 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 25$$

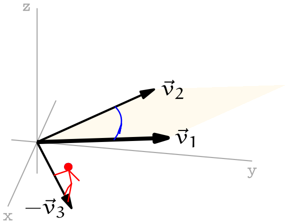
This is the *right hand rule*: place your right hand's pinky finger on the spanned plane so that your fingers curl from $\vec{v}_1$ to $\vec{v}_2$. Then vectors on the side of the plane with your thumb define positive-sized boxes.

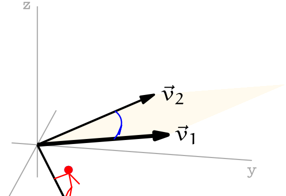


A vector on the plane's other side, such as $-\vec{v}_3$, will have the same arc from $\vec{v}_1$ to $\vec{v}_2$ look clockwise and will define a negative-sized box.

$$\begin{vmatrix} 1 & -2 & 0 \\ 4 & 3 & 1 \\ 1 & 1 & -2 \end{vmatrix} = -25$$

These pictures start above the plane, the perspective of the prior slide. From below, to a person standing on $-\vec{v}_3$, the arc is clockwise.





Determinants are multiplicative

1.5 *Theorem* A transformation $t\colon \mathbb{R}^n \to \mathbb{R}^n$ changes the size of all boxes by the same factor, namely, the size of the image of a box $|t(S)|$ is $|T|$ times the size of the box $|S|$, where T is the matrix representing t with respect to the standard basis.

That is, the determinant of a product is the product of the determinants $|TS| = |T| \cdot |S|$.

Proof First consider the case that T is singular and thus does not have an inverse. Observe that if TS is invertible then there is an M such that $(TS)M = I$, so $T(SM) = I$, and so T is invertible. The contrapositive of that observation is that if T is not invertible then neither is TS — if $|T| = 0$ then $|TS| = 0$.

Now consider the case that T is nonsingular. Any nonsingular matrix factors into a product of elementary matrices $T = E_1 E_2 \cdots E_r$. To finish this argument we will verify that $|ES| = |E| \cdot |S|$ for all matrices S and elementary matrices E. The result will then follow because $|TS| = |E_1 \cdots E_r S| = |E_1| \cdots |E_r| \cdot |S| = |E_1 \cdots E_r| \cdot |S| = |T| \cdot |S|$.

There are three types of elementary matrix. We will cover the $M_i(k)$ case; the $P_{i,j}$ and $C_{i,j}(k)$ checks are similar. The matrix $M_i(k)S$ equals S except that row i is multiplied by k. The third condition of determinant functions then gives that $|M_i(k)S| = k \cdot |S|$. But $|M_i(k)| = k$, again by the third condition because $M_i(k)$ is derived from the identity by multiplication of row i by k. Thus $|ES| = |E| \cdot |S|$ holds for $E = M_i(k)$. QED

Example The transformation $t_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates all vectors through a counterclockwise angle θ is represented by this matrix.

$$T_\theta = \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t_\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

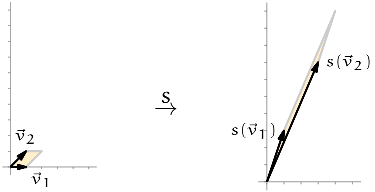
Observe that t_θ doesn't change the size of any boxes, it just rotates the entire box as a rigid whole. Note that $|T_\theta| = 1$.

Example The linear transformation $s \colon \mathbb{R}^2 \to \mathbb{R}^2$ represented with respect to the standard basis by this matrix

$$S = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

will, by the theorem, change the size of a box by a factor of $|S| = -2$. Here is s acting on a typical box.

The box defined by the two vectors $\vec{v}_1 = \binom{1}{0}$ and $\vec{v}_2 = \binom{1}{1}$ is transformed by s to the box defined by the two vectors $s(\vec{v}_1) = \binom{1}{3}$ and $s(\vec{v}_2) = \binom{3}{7}$.



Note the change in orientation, matching that the determinant is negative.

The two sizes are easy.

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1 & 3 \\ 3 & 7 \end{vmatrix} = -2$$

Determinant of the inverse

1.7 *Corollary* If a matrix is invertible then the determinant of its inverse is the inverse of its determinant $|T^{-1}| = 1/|T|$.

Proof $1 = |I| = |TT^{-1}| = |T| \cdot |T^{-1}|$ QED

Example These matrices are inverse.

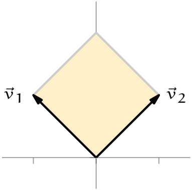
$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \qquad \begin{vmatrix} -2 & 1 \\ 3/2 & -1/2 \end{vmatrix} = -1/2$$

Volume

1.3 *Definition* The *volume* of a box is the absolute value of the determinant of a matrix with those vectors as columns.

Example The box formed by the vectors

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$$



gives this determinant

$$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

so its volume is 2.

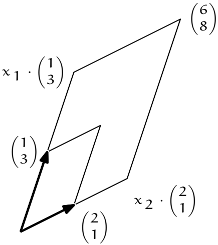
Cramer's Rule

Geometric interpretation of linear systems

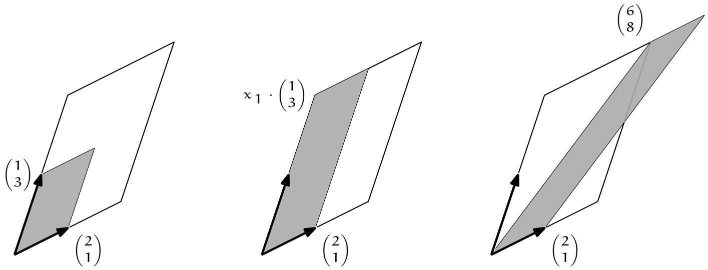
A linear system is equivalent to a linear relationship among vectors.

$$\begin{aligned} x_1 + 2x_2 &= 6 \\ 3x_1 + x_2 &= 8 \end{aligned} \qquad \Longleftrightarrow \qquad x_1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

This drawing restates the algebraic question of finding the solution of a linear system into geometric terms: by what factors x_1 and x_2 must we dilate the sides of the starting parallelogram so that it will fill the other one?



Consider expanding only one side of the parallelogram. Compare the sizes of these shaded boxes.



Taken together we have this.

$$x_1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 \cdot 1 & 2 \\ x_1 \cdot 3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 \cdot 1 + x_2 \cdot 2 & 2 \\ x_1 \cdot 3 + x_2 \cdot 1 & 1 \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 8 & 1 \end{vmatrix}$$

(The last equality is by the first condition in the definition of determinants, applied to columns.)

Solving gives the value of one of the variables.

$$x_1 = \frac{\begin{vmatrix} 6 & 2 \\ 8 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = \frac{-10}{-5} = 2$$

The symmetric argument for the other side gives the other value.

$$x_2 = \frac{\begin{vmatrix} 1 & 6 \\ 3 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = 2$$

Cramer's Rule

Theorem Let A be an $n \times n$ matrix with a nonzero determinant, let $\vec{b}$ be an n-tall column vector, and consider the linear system $A\vec{x} = \vec{b}$. For any $i \in [1, \ldots, n]$ let B_i be the matrix obtained by substituting $\vec{b}$ for column i of A. Then the value of the i-th unknown is $x_i = |B_i|/|A|$.

Of course, if the matrix has a zero determinant then the system does not have a unique solution.

Example This system

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4 \\ x_1 + 3x_2 \phantom{{}-x_3} &= 2 \\ x_2 - 5x_3 &= 0 \end{aligned}$$

is

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ 0 & 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

and

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ 0 & 1 & -5 \end{vmatrix} = -26 \qquad |B_2| = \begin{vmatrix} 2 & 4 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & -5 \end{vmatrix} = 0$$

so $x_2 = 0/-26 = 0$.

A caution

Cramer's Rule is an interesting application of the geometry. And, it allows us to mentally solve small systems with simple numbers. But don't use it for systems having many variables; taking a determinant of a general large matrix is very slow.