Three.V Change of Basis

Linear Algebra

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Changing representations of vectors

Coordinates vary with the basis

Consider this vector $\vec{v} \in \mathbb{R}^3$ and bases for the space.

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathcal{E}_3 = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle \quad B = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rangle$$

With respect to the different bases, the coordinates of $\vec{v}$ are different.

$$\mathrm{Rep}_{\mathcal{E}_3}(\vec{v}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathrm{Rep}_{B}(\vec{v}) = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

In this section we will see how to convert the representation of a vector with respect to a first basis to its representation with respect to a second.

Change of basis matrix

Think of translating from $\mathrm{Rep}_{\mathrm{B}}(\vec{v})$ to $\mathrm{Rep}_{\mathrm{D}}(\vec{v})$ as holding the vector constant. This is the arrow diagram.

$$\begin{array}{c} V_{wrt\ \mathrm{B}} \\ {\scriptstyle \mathrm{id}}\big\downarrow \\ V_{wrt\ \mathrm{D}} \end{array}$$

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1.1 *Definition* The *change of basis matrix* for bases $\mathrm{B}, \mathrm{D} \subset V$ is the representation of the identity map $\mathrm{id}\colon V \to V$ with respect to those bases.

$$\mathrm{Rep}_{\mathrm{B},\mathrm{D}}(\mathrm{id}) = \begin{pmatrix} \vdots & & \vdots \\ \mathrm{Rep}_{\mathrm{D}}(\vec{\beta}_1) & \cdots & \mathrm{Rep}_{\mathrm{D}}(\vec{\beta}_n) \\ \vdots & & \vdots \end{pmatrix}$$

1.3 *Lemma* To convert from the representation of a vector $\vec{v}$ with respect to B to its representation with respect to D use the change of basis matrix.

$$\mathrm{Rep}_{B,D}(\mathrm{id})\,\mathrm{Rep}_{B}(\vec{v}) = \mathrm{Rep}_{D}(\vec{v})$$

Conversely, if left-multiplication by a matrix changes bases $M \cdot \mathrm{Rep}_{B}(\vec{v}) = \mathrm{Rep}_{D}(\vec{v})$ then M is a change of basis matrix.

Proof The first sentence holds because matrix-vector multiplication represents a map application and so $\mathrm{Rep}_{B,D}(\mathrm{id}) \cdot \mathrm{Rep}_{B}(\vec{v}) = \mathrm{Rep}_{D}(\,\mathrm{id}(\vec{v})\,) = \mathrm{Rep}_{D}(\vec{v})$ for each $\vec{v}$. For the second sentence, with respect to B, D the matrix M represents a linear map whose action is to map each vector to itself, and is therefore the identity map.

QED

Example To change a representation of a member of $\mathcal{P}_2$ from being with respect to $B = \langle 1, 1+x, 1+x+x^2 \rangle$ to being with respect to $D = \langle x^2-1, x, x^2+1 \rangle$, compute $\mathrm{Rep}_{B,D}(\mathrm{id})$. The identity map acting on the elements of B has no effect. Represent those elements with respect to D.

$$\mathrm{Rep}_D(1) = \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix} \quad \mathrm{Rep}_D(1+x) = \begin{pmatrix} -1/2 \\ 1 \\ 1/2 \end{pmatrix} \quad \mathrm{Rep}_D(1+x+x^2) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The change of basis matrix is the concatenation of those.

$$\mathrm{Rep}_{B,D}(\mathrm{id}) = \begin{pmatrix} -1/2 & -1/2 & 0 \\ 0 & 1 & 1 \\ 1/2 & 1/2 & 1 \end{pmatrix}$$

For example, we can translate the representation of $\vec{v} = 2 - x + 3x^2$.

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} \qquad \mathrm{Rep}_D(\vec{v}) = \begin{pmatrix} 1/2 \\ -1 \\ 5/2 \end{pmatrix} = \begin{pmatrix} -1/2 & -1/2 & 0 \\ 0 & 1 & 1 \\ 1/2 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$$

1.5 *Lemma* A matrix changes bases if and only if it is nonsingular.

Proof For the 'only if' direction, if left-multiplication by a matrix changes bases then the matrix represents an invertible function, simply because we can invert the function by changing the bases back. Because it represents a function that is invertible, the matrix itself is invertible, and so is nonsingular.

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For 'if' we will show that any nonsingular matrix M performs a change of basis operation from any given starting basis B (having n vectors, where the matrix is $n \times n$) to some ending basis.

If the matrix is the identity I then the statement is obvious. Otherwise because the matrix is nonsingular Corollary IV.3.23 says there are elementary reduction matrices such that $R_r \cdots R_1 \cdot M = I$ with $r \geqslant 1$. Elementary matrices are invertible and their inverses are also elementary so multiplying both sides of that equation from the left by ${R_r}^{-1}$, then by ${R_{r-1}}^{-1}$, etc., gives M as a product of elementary matrices $M = {R_1}^{-1} \cdots {R_r}^{-1}$.

We will be done if we show that elementary matrices change a given basis to another basis, since then ${R_r}^{-1}$ changes B to some other basis B_r and ${R_{r-1}}^{-1}$ changes B_r to some B_{r-1}, etc. We will cover the three types of elementary matrices separately; recall the notation for the three.

$$M_i(k)\begin{pmatrix} c_1 \\ \vdots \\ c_i \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ kc_i \\ \vdots \\ c_n \end{pmatrix} \quad P_{i,j}\begin{pmatrix} c_1 \\ \vdots \\ c_i \\ \vdots \\ c_j \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_j \\ \vdots \\ c_i \\ \vdots \\ c_n \end{pmatrix} \quad C_{i,j}(k)\begin{pmatrix} c_1 \\ \vdots \\ c_i \\ \vdots \\ c_j \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_i \\ \vdots \\ kc_i + c_j \\ \vdots \\ c_n \end{pmatrix}$$

Applying a row-multiplication matrix $M_i(k)$ changes a representation with respect to $\langle\vec{\beta}_1,\ldots,\vec{\beta}_i,\ldots,\vec{\beta}_n\rangle$ to one with respect to $\langle\vec{\beta}_1,\ldots,(1/k)\vec{\beta}_i,\ldots,\vec{\beta}_n\rangle$.

$$\begin{aligned}\vec{v}=c_1\cdot\vec{\beta}_1+\cdots+c_i\cdot\vec{\beta}_i+\cdots+c_n\cdot\vec{\beta}_n\\ \mapsto\; c_1\cdot\vec{\beta}_1+\cdots+kc_i\cdot(1/k)\vec{\beta}_i+\cdots+c_n\cdot\vec{\beta}_n=\vec{v}\end{aligned}$$

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Similarly, left-multiplication by a row-swap matrix $P_{i,j}$ changes a representation with respect to the basis $\langle\vec{\beta}_1,\dots,\vec{\beta}_i,\dots,\vec{\beta}_j,\dots,\vec{\beta}_n\rangle$ into one with respect to this basis $\langle\vec{\beta}_1,\dots,\vec{\beta}_j,\dots,\vec{\beta}_i,\dots,\vec{\beta}_n\rangle$.

$$\begin{aligned}\vec{v}=c_1\cdot\vec{\beta}_1+\cdots+c_i\cdot\vec{\beta}_i+\cdots+c_j\vec{\beta}_j+\cdots+c_n\cdot\vec{\beta}_n\\ \mapsto\ c_1\cdot\vec{\beta}_1+\cdots+c_j\cdot\vec{\beta}_j+\cdots+c_i\cdot\vec{\beta}_i+\cdots+c_n\cdot\vec{\beta}_n=\vec{v}\end{aligned}$$

And, a representation with respect to $\langle\vec{\beta}_1,\ldots,\vec{\beta}_i,\ldots,\vec{\beta}_j,\ldots,\vec{\beta}_n\rangle$ changes via left-multiplication by a row-combination matrix $C_{i,j}(k)$ into a representation with respect to $\langle\vec{\beta}_1,\ldots,\vec{\beta}_i-k\vec{\beta}_j,\ldots,\vec{\beta}_j,\ldots,\vec{\beta}_n\rangle$

$$\begin{aligned}
&\vec{v}=c_1\cdot\vec{\beta}_1+\cdots+c_i\cdot\vec{\beta}_i+c_j\vec{\beta}_j+\cdots+c_n\cdot\vec{\beta}_n\\
&\mapsto\ c_1\cdot\vec{\beta}_1+\cdots+c_i\cdot(\vec{\beta}_i-k\vec{\beta}_j)+\cdots+(kc_i+c_j)\cdot\vec{\beta}_j+\cdots+c_n\cdot\vec{\beta}_n=\vec{v}
\end{aligned}$$

(the definition of $C_{i,j}(k)$ specifies that $i\neq j$ and $k\neq 0$). QED

1.6 *Corollary* A matrix is nonsingular if and only if it represents the identity map with respect to some pair of bases.

Changing map representations

The natural next step is to see how to convert $\mathrm{Rep}_{B,D}(h)$ to $\mathrm{Rep}_{\hat{B},\hat{D}}(h)$. Here is the arrow diagram.

$$\begin{tikzcd}
V_{wrt\ B} \arrow[r, "h", "H"'] \arrow[d, "id"'] & W_{wrt\ D} \arrow[d, "id"'] \\
V_{wrt\ \hat{B}} \arrow[r, "h", "\hat{H}"'] & W_{wrt\ \hat{D}}
\end{tikzcd}$$

To move from the lower-left to the lower-right we can either go straight over, or else up to V_B then over to W_D and then down. So we can calculate $\hat{H} = \mathrm{Rep}_{\hat{B},\hat{D}}(h)$ either by directly using $\hat{B}$ and $\hat{D}$, or else by first changing bases with $\mathrm{Rep}_{\hat{B},B}(\mathrm{id})$ then multiplying by $H = \mathrm{Rep}_{B,D}(h)$ and then changing bases with $\mathrm{Rep}_{D,\hat{D}}(\mathrm{id})$.

Theorem To convert from the matrix H representing a map h with respect to B, D to the matrix $\hat{H}$ representing it with respect to $\hat{B}, \hat{D}$ use this formula.

$$\hat{H} = \mathrm{Rep}_{D,\hat{D}}(\mathrm{id}) \cdot H \cdot \mathrm{Rep}_{\hat{B},B}(\mathrm{id}) \tag{$*$}$$

Proof This is evident from the diagram. QED

Example Consider the derivative map $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_2$ as well as $B = \langle 1, 1+x, 1+x+x^2 \rangle, D = \langle 1+x^2, x, 1-x^2 \rangle$ and $\hat{B} = \langle 1, x, x^2 \rangle, \hat{D} = \langle 1+x, x+x^2, 1+x^2 \rangle$.

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We can find H and $\hat{H}$ using the methods we have already seen.

$$\mathrm{Rep}_{B,D}(d/dx) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \quad \mathrm{Rep}_{\hat{B},\hat{D}}(d/dx) = \begin{pmatrix} 0 & 1/2 & 1 \\ 0 & -1/2 & 1 \\ 0 & 1/2 & -1 \end{pmatrix}$$

These do the base changes.

$$\mathrm{Rep}_{\hat{B},B}(\mathrm{id}) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathrm{Rep}_{D,\hat{D}}(\mathrm{id}) = \begin{pmatrix} 0 & 1/2 & 1 \\ 0 & 1/2 & -1 \\ 1 & -1/2 & 0 \end{pmatrix}$$

We can check this case of equation $(*)$ by multiplying through.

$$\begin{aligned} \mathrm{Rep}_{\hat{B},\hat{D}}(d/dx) &= \mathrm{Rep}_{D,\hat{D}}(\mathrm{id}) \cdot \mathrm{Rep}_{B,D}(d/dx) \cdot \mathrm{Rep}_{\hat{B},B}(\mathrm{id}) \\ &= \begin{pmatrix} 0 & 1/2 & 1 \\ 0 & 1/2 & -1 \\ 1 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Example The map $t_{\pi/6}\colon \mathbb{R}^2 \to \mathbb{R}^2$ rotating vectors counterclockwise by $\pi/6$ radians has this representation with respect to the standard bases.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t_{\pi/6}) = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

We can translate to the representatation with respect to

$$B = D = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle$$

using the change of basis matrices. Here is the diagram, specialized for this case.

$$\begin{array}{ccc}
\mathbb{R}^2_{wrt\ \mathcal{E}_2} & \xrightarrow[\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t_{\pi/6})]{t_{\pi/6}} & \mathbb{R}^2_{wrt\ \mathcal{E}_2} \\
\mathrm{id}\big\downarrow & & \mathrm{id}\big\downarrow \\
\mathbb{R}^2_{wrt\ B} & \xrightarrow[\hat{H}]{t_{\pi/6}} & \mathbb{R}^2_{wrt\ D}
\end{array}$$

To get $\hat{H}$ we move up from the upper left, across, and then down.

With respect to the standard basis real vectors represent themselves, so the matrix representing moving up is easy.

$$\mathrm{Rep}_{B,\mathcal{E}_2}(\mathrm{id}) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The matrix for moving down is the inverse of the prior one.

$$\mathrm{Rep}_{\mathcal{E}_2,D}(\mathrm{id}) = (\mathrm{Rep}_{D,\mathcal{E}_2}(\mathrm{id}))^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{-1}{2} \cdot \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

Thus we have this.

$$\begin{aligned} \mathrm{Rep}_{B,D}(t_{\pi/6}) &= \frac{-1}{2} \cdot \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} \end{aligned}$$

2.4 *Definition* Same-sized matrices H and $\hat{H}$ are *matrix equivalent* if there are nonsingular matrices P and Q such that $\hat{H} = PHQ$.

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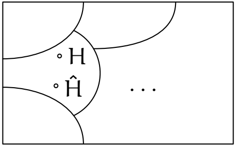
Exercise 24 checks that matrix equivalence is an equivalence relation. Thus it partitions the set of matrices into matrix equivalence classes.

All matrices:



H matrix equivalent to $\hat{H}$

Canonical form for matrix equivalence

2.7 *Theorem* Any $m\times n$ matrix of rank k is matrix equivalent to the $m\times n$ matrix that is all zeros except that the first k diagonal entries are ones.

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}$$

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This is a *block partial-identity* form.

$$\left(\begin{array}{c|c} I & Z \\ \hline Z & Z \end{array}\right)$$

For an example, recall first that any nonsingular matrix M can be factored into a product $M = R_1 \cdots R_r$ where each R_t is an elementary reduction matrix $C_{i,j}(k)$, or $M_i(k)$, or $P_{i,j}$.

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Recall also that from the left matrices operate on rows

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 0 & -3 \end{pmatrix}$$

while from the right they act on columns: this does $-4 \cdot \text{col}_3 + \text{col}_1$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -11 & 2 & 3 \\ -20 & 5 & 6 \\ -29 & 8 & 9 \end{pmatrix}$$

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Example This matrix has rank 2.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

We will find P and Q to get the 3×3 canonical matrix of rank 2.

Row operations produce echelon form.

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\;\xrightarrow[-7\rho_1+\rho_3]{-4\rho_1+\rho_2}\;\xrightarrow{-2\rho_2+\rho_3}\;\xrightarrow{-(1/3)\rho_2}\;\begin{pmatrix}1&2&3\\0&1&2\\0&0&0\end{pmatrix}$$

Multiplying the reduction matrices gives P; note the right to left order.

$$\begin{pmatrix}1&0&0\\0&-1/3&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&-2&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-7&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-4&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\4/3&-1/3&0\\1&-2&1\end{pmatrix}$$

So far we have this.

$$PHQ=\begin{pmatrix}1&0&0\\4/3&-1/3&0\\1&-2&1\end{pmatrix}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}Q=\begin{pmatrix}1&2&3\\0&1&2\\0&0&0\end{pmatrix}Q$$

Column operations produce the block partial identity.

$$\begin{pmatrix}1&2&3\\0&1&2\\0&0&0\end{pmatrix}\quad\xrightarrow{-2\mathrm{col}_2+\mathrm{col}_3}\quad\xrightarrow{-2\mathrm{col}_1+\mathrm{col}_2}\quad\xrightarrow{\mathrm{col}_1+\mathrm{col}_3}\quad\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

Combine the reduction matrices to get Q. In contrast with the construction of P, here they come left to right.

$$\begin{pmatrix}1&0&0\\0&1&-2\\0&0&1\end{pmatrix}\begin{pmatrix}1&-2&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&-2&1\\0&1&-2\\0&0&1\end{pmatrix}$$

In sum, we have this equation.

$$PHQ=\begin{pmatrix}1&0&0\\4/3&-1/3&0\\1&-2&1\end{pmatrix}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}1&-2&1\\0&1&-2\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

Proof Gauss-Jordan reduce the given matrix and combine all the row reduction matrices to make P. Then use the leading entries to do column reduction and finish by swapping the columns to put the leading ones on the diagonal. Combine the column reduction matrices into Q. QED

Action of a canonical form matrix

This kind of matrix is has an easy to understand effect.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

It is a projection — the map $t\colon \mathbb{R}^3 \to \mathbb{R}^3$ represented with respect to the standard bases by this block partial identity matrix is projection from three space to the xy plane.

Matrix equivalence is characterized by rank

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Proof Two same-sized matrices with the same rank are equivalent to the same block partial-identity matrix. QED