

Three.VI Projection

Linear Algebra

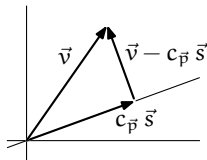
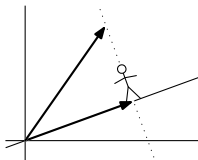
Jim Hefferon

<http://joshua.smcvt.edu/linearalgebra>

Orthogonal Projection Into a Line

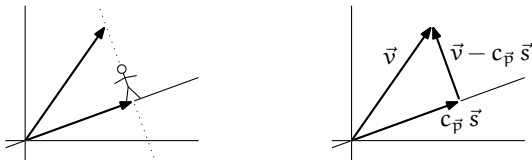
Project a vector into a line

This shows a figure walking out on the line to a point \vec{p} such that the tip of \vec{v} is directly above them, where “above” does not mean parallel to the y-axis but instead means orthogonal to the line.



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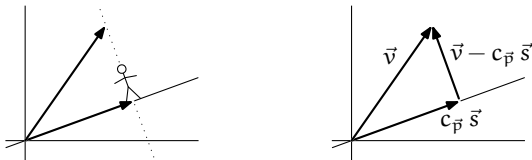
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Since the line is the span of some vector $\ell = \{c \cdot \vec{s} \mid c \in \mathbb{R}\}$, we have a coefficient $c_{\vec{p}}$ with the property that $\vec{v} - c_{\vec{p}} \vec{s}$ is orthogonal to $c_{\vec{p}} \vec{s}$.

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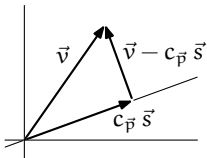
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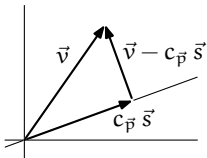
To solve for this coefficient, observe that because $\vec{v} - c_{\vec{p}} \vec{s}$ is orthogonal to a scalar multiple of \vec{s} , it must be orthogonal to \vec{s} itself. Then $(\vec{v} - c_{\vec{p}} \vec{s}) \cdot \vec{s} = 0$ gives that $c_{\vec{p}} = \vec{v} \cdot \vec{s} / \vec{s} \cdot \vec{s}$.

We have decomposed \vec{v} into two parts $\vec{v} = (c_{\vec{p}}\vec{s}) + (\vec{v} - c_{\vec{p}}\vec{s})$.



Intuitively, some of \vec{v} lies with the line and that gives the first part $c_{\vec{p}}\vec{s}$. The part of \vec{v} that lies with a line orthogonal to ℓ is $\vec{v} - c_{\vec{p}}\vec{s}$. What's compelling about pairing these two parts is that they don't interact, in that the projection of one into the line spanned by the other is the zero vector.

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Note: We have not given a definition of 'angle' in spaces other than \mathbb{R}^n 's, so we will stick here to those spaces. Extending the definitions to other spaces is perfectly possible but we don't need them here.

1.1 *Definition* The *orthogonal projection of \vec{v} into the line spanned by a nonzero \vec{s}* is this vector.

$$\text{proj}_{[\vec{s}]}(\vec{v}) = \frac{\vec{v} \cdot \vec{s}}{\vec{s} \cdot \vec{s}} \cdot \vec{s}$$

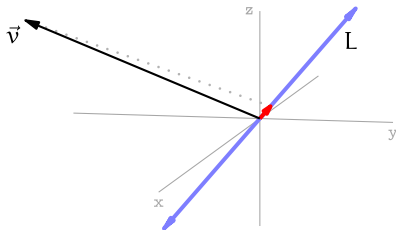
Example The projection of this \mathbb{R}^3 vector into the line

$$\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad L = \{c \cdot \vec{s} = c \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \mid c \in \mathbb{R}\}$$

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$$\frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/6 \\ 1/6 \end{pmatrix}$$

Because \vec{v} is nearly orthogonal to the line L

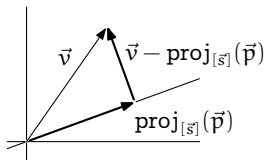


only a small part of \vec{v} lies with the direction of that line, so the projected-to red vector $\text{proj}_{[\vec{s}]}(\vec{v})$ is quite short: ($|\vec{v}| = \sqrt{6} \approx 2.45$ while $|\text{proj}_{[\vec{s}]}(\vec{v})| = \sqrt{1/6} \approx 0.41$).

Gram-Schmidt Orthogonalization

Mutually orthogonal vectors

The prior subsection suggests that projecting a vector \vec{v} into the line spanned by \vec{s} decomposes \vec{v} into two parts, a part with the line and a part orthogonal to that.

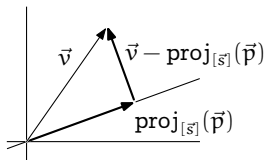


$$\vec{v} = \text{proj}_{[\vec{s}]}(\vec{v}) + \left(\vec{v} - \text{proj}_{[\vec{s}]}(\vec{v}) \right)$$

Because these are orthogonal they are in some sense non-interacting. Here we will develop that.

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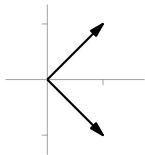
2.1 Definition Vectors $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ are *mutually orthogonal* when any two are orthogonal: if $i \neq j$ then the dot product $\vec{v}_i \cdot \vec{v}_j$ is zero.

Example The vectors of the standard basis $\mathcal{E}_3 \subset \mathbb{R}^3$ are mutually orthogonal.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Example These two vectors in \mathbb{R}^2 are mutually orthogonal.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



The next result makes ‘non-interacting’ precise.

2.2 *Theorem* If the vectors in a set $\{\vec{v}_1, \dots, \vec{v}_k\} \subset \mathbb{R}^n$ are mutually orthogonal and nonzero then that set is linearly independent.

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Proof Consider $\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$. For $i \in \{1, \dots, k\}$, taking the dot product of \vec{v}_i with both sides of the equation $\vec{v}_i \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k) = \vec{v}_i \cdot \vec{0}$, which gives $c_i \cdot (\vec{v}_i \cdot \vec{v}_i) = 0$, shows that $c_i = 0$ since $\vec{v}_i \neq \vec{0}$. QED

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2.5 *Definition* An *orthogonal basis* for a vector space is a basis of mutually orthogonal vectors.

2.7 *Theorem* If $\langle \vec{\beta}_1, \dots, \vec{\beta}_k \rangle$ is a basis for a subspace of \mathbb{R}^n then the vectors

$$\vec{\kappa}_1 = \vec{\beta}_1$$

$$\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)$$

$$\vec{\kappa}_3 = \vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)$$

$$\vdots$$

$$\vec{\kappa}_k = \vec{\beta}_k - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_k) - \dots - \text{proj}_{[\vec{\kappa}_{k-1}]}(\vec{\beta}_k)$$

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Proof We will use induction to check that each $\vec{\kappa}_i$ is nonzero, is in the span of $\langle \vec{\beta}_1, \dots, \vec{\beta}_i \rangle$, and is orthogonal to all preceding vectors $\vec{\kappa}_1 \cdot \vec{\kappa}_i = \dots = \vec{\kappa}_{i-1} \cdot \vec{\kappa}_i = 0$. Then Corollary 2.3 gives that $\langle \vec{\kappa}_1, \dots, \vec{\kappa}_k \rangle$ is a basis for the same space as is the starting basis.

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We shall only cover the cases up to $i = 3$, to give the sense of the argument. The full argument is Exercise 28 .

The $i = 1$ case is trivial; taking $\vec{\kappa}_1$ to be $\vec{\beta}_1$ makes it a nonzero vector since $\vec{\beta}_1$ is a member of a basis, it is obviously in the span of $\langle \vec{\beta}_1 \rangle$, and the ‘orthogonal to all preceding vectors’ condition is satisfied vacuously.

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In the $i = 2$ case the expansion

$$\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2) = \vec{\beta}_2 - \frac{\vec{\beta}_2 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\kappa}_1 = \vec{\beta}_2 - \frac{\vec{\beta}_2 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\beta}_1$$

shows that $\vec{\kappa}_2 \neq \vec{0}$ or else this would be a non-trivial linear dependence among the $\vec{\beta}$ ’s (it is nontrivial because the coefficient of $\vec{\beta}_2$ is 1). It also shows that $\vec{\kappa}_2$ is in the span of $\langle \vec{\beta}_1, \vec{\beta}_2 \rangle$. And, $\vec{\kappa}_2$ is orthogonal to the only preceding vector

$$\vec{\kappa}_1 \cdot \vec{\kappa}_2 = \vec{\kappa}_1 \cdot (\vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)) = 0$$

because this projection is orthogonal.

The $i = 3$ case is the same as the $i = 2$ case except for one detail. As in the $i = 2$ case, expand the definition.

$$\begin{aligned}\vec{\kappa}_3 &= \vec{\beta}_3 - \frac{\vec{\beta}_3 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\kappa}_1 - \frac{\vec{\beta}_3 \cdot \vec{\kappa}_2}{\vec{\kappa}_2 \cdot \vec{\kappa}_2} \cdot \vec{\kappa}_2 \\ &= \vec{\beta}_3 - \frac{\vec{\beta}_3 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\beta}_1 - \frac{\vec{\beta}_3 \cdot \vec{\kappa}_2}{\vec{\kappa}_2 \cdot \vec{\kappa}_2} \cdot \left(\vec{\beta}_2 - \frac{\vec{\beta}_2 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\beta}_1 \right)\end{aligned}$$

By the first line $\vec{\kappa}_3 \neq \vec{0}$, since $\vec{\beta}_3$ isn't in the span $[\vec{\beta}_1, \vec{\beta}_2]$ and therefore by the inductive hypothesis it isn't in the span $[\vec{\kappa}_1, \vec{\kappa}_2]$. By the second line $\vec{\kappa}_3$ is in the span of the first three $\vec{\beta}$'s. Finally, the calculation below shows that $\vec{\kappa}_3$ is orthogonal to $\vec{\kappa}_1$.

$$\begin{aligned}
\vec{\kappa}_1 \cdot \vec{\kappa}_3 &= \vec{\kappa}_1 \cdot (\vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)) \\
&= \vec{\kappa}_1 \cdot (\vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3)) - \vec{\kappa}_1 \cdot \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3) \\
&= 0
\end{aligned}$$

(Here is the difference with the $i = 2$ case: as happened for $i = 2$ the first term is 0 because this projection is orthogonal, but here the second term in the second line is 0 because $\vec{\kappa}_1$ is orthogonal to $\vec{\kappa}_2$ and so is orthogonal to any vector in the line spanned by $\vec{\kappa}_2$.) A similar check shows that $\vec{\kappa}_3$ is also orthogonal to $\vec{\kappa}_2$. QED

Example This basis for \mathbb{R}^2

$$B = \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\rangle$$

does not have orthogonal vectors. To derive from it a basis $K = \langle \vec{\kappa}_1, \vec{\kappa}_2 \rangle$ that is orthogonal, start by taking the first vector unchanged.

$$\vec{\kappa}_1 = \vec{\beta}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\vec{\kappa}_2$ take the part of $\vec{\beta}_2$ that does not lie with $\vec{\kappa}_1$.

$$\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2/5 \\ 1/5 \end{pmatrix}$$

Note that $\vec{\kappa}_1$ and $\vec{\kappa}_2$ are indeed orthogonal.

Example This is a basis for \mathbb{R}^3 .

$$B = \left\langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \right\rangle$$

Start the orthogonal basis with $\vec{\beta}_1$.

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As in the prior slide, the next step is $\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)$.

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}$$

The third step is $\vec{\kappa}_3 = \vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)$.

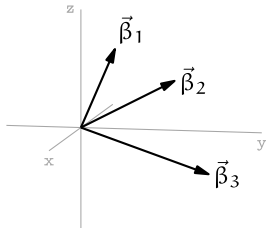
$$\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}}{\begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 4/3 \\ -4/3 \end{pmatrix}$$

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The members of B are at odd angles but the members of K are mutually orthogonal.

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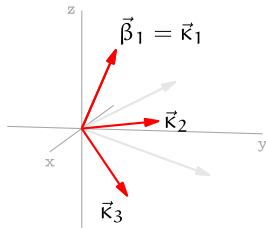


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$$K = \left\langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4/3 \\ 4/3 \\ -4/3 \end{pmatrix} \right\rangle$$

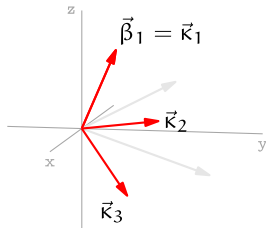


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$$\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}}{\begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 4/3 \\ -4/3 \end{pmatrix}$$

The members of B are at odd angles but the members of K are mutually orthogonal.

$$K = \left\langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4/3 \\ 4/3 \\ -4/3 \end{pmatrix} \right\rangle$$



We could go on to make this basis even more like \mathcal{E}_3 by normalizing all of its members to have length 1, making an *orthonormal* basis.