

## One.II Linear Geometry

*Linear Algebra* edition four  
Jim Hefferon

<http://joshua.smcvt.edu/linearalgebra>

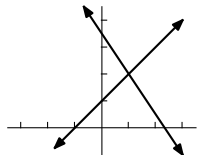
## Geometry

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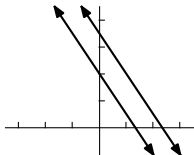
*Unique solution*



$$3x + 2y = 7$$

$$x - y = -1$$

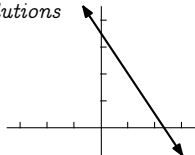
*No solutions*



$$3x + 2y = 7$$

$$3x + 2y = 4$$

*Infinitely many solutions*



$$3x + 2y = 7$$

$$6x + 4y = 14$$

This is a nice restatement of the possibilities; the geometry gives us insight into what can happen with linear systems.

## Vectors in space

## Vectors

A *vector* is an object consisting of a magnitude and a direction.



A vector can model a displacement.

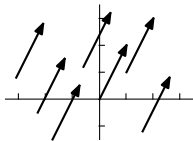
## Vectors

A *vector* is an object consisting of a magnitude and a direction.



A vector can model a displacement.

Two vectors with the same magnitude and same direction, such as all of these, are equal.



For instance, each of the above could model a displacement of one over and two up.

Denote the vector that extends from  $(a_1, a_2)$  to  $(b_1, b_2)$  by

$$\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

so the “one over, two up” vector would be written in this way.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

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We often picture a vector

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

as starting at the origin. From there  $\vec{v}$  extends to  $(v_1, v_2)$  and we may refer to it as “the point  $\vec{v}$ ” so that we may call each of these  $\mathbb{R}^2$ .

$$\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\} \quad \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$$



These definitions extend to higher dimensions. The vector that starts at  $(a_1, \dots, a_n)$  and ends at  $(b_1, \dots, b_n)$  is represented by this column

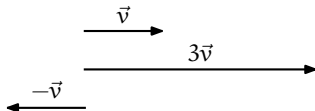
$$\begin{pmatrix} b_1 - a_1 \\ \vdots \\ b_n - a_n \end{pmatrix}$$

and two vectors are equal if they have the same representation. Also, we aren't too careful about distinguishing between a point and the vector which, when it starts at the origin, ends at that point.

$$\mathbb{R}^n = \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mid v_1, \dots, v_n \in \mathbb{R} \right\}$$

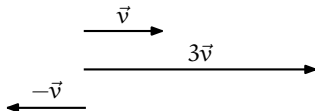
## Vector operations

Scalar multiplication makes a vector longer or shorter, including possibly flipping it around.

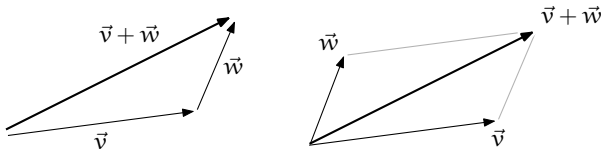


## Vector operations

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Where  $\vec{v}$  and  $\vec{w}$  represent displacements, the vector sum  $\vec{v} + \vec{w}$  represents those displacements combined.

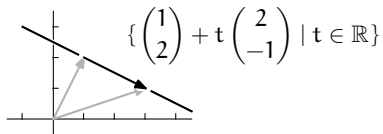


The second drawing shows the *parallelogram rule* for vector addition.

## Lines

The line in  $\mathbb{R}^2$  through  $(1, 2)$  and  $(3, 1)$  is comprised of the vectors in this set

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

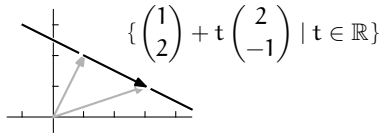


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(that is, it is comprised of the endpoints of those vectors). The vector associated with the parameter  $t$

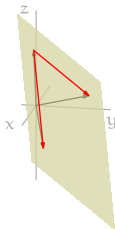
$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

is a *direction vector* for the line. Lines in higher dimensions work the same way.

## Planes

The plane in  $\mathbb{R}^3$  through the points  $(1, 0, 5)$ ,  $(2, 1, -3)$ , and  $(-2, 4, 0.5)$  consists of (endpoints of) the vectors in this set.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -8 \end{pmatrix} + s \begin{pmatrix} -3 \\ 4 \\ -4.5 \end{pmatrix} \mid t, s \in \mathbb{R} \right\}$$



The column vectors associated with a parameter, shown here in red,

$$\begin{pmatrix} 1 \\ 1 \\ -8 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} \quad \begin{pmatrix} -3 \\ 4 \\ -4.5 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0.5 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

have their whole body in the plane, not just their endpoint.

A set of the form  $\{\vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k \mid t_1, \dots, t_k \in \mathbb{R}\}$  where  $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$  and  $k \leq n$  is a *k-dimensional linear surface* (or *k-flat*).

Length and angle measures

## Length

2.1 *Definition* The *length* of a vector  $\vec{v} \in \mathbb{R}^n$  is the square root of the sum of the squares of its components.

$$|\vec{v}| = \sqrt{v_1^2 + \cdots + v_n^2}$$

*Example* The length of

$$\vec{v} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$$

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For any nonzero vector  $\vec{v}$ , the length one vector with the same direction is  $\vec{v}/|\vec{v}|$ . We say that this *normalizes*  $\vec{v}$  to unit length.

$$\frac{\vec{v}}{|\vec{v}|} = \begin{pmatrix} -1/\sqrt{14} \\ -2/\sqrt{14} \\ -3/\sqrt{14} \end{pmatrix}$$

Note that  $\sqrt{(-1/\sqrt{14})^2 + (-2/\sqrt{14})^2 + (-3/\sqrt{14})^2}$  equals 1.

## Dot product

2.3 *Definition* The *dot product* (or *inner product* or *scalar product*) of two  $n$ -component real vectors is the linear combination of their components.

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

*Example* The dot product of two vectors

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} = 3 - 3 - 4 = -4$$

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The dot product of a vector with itself  $\vec{v} \cdot \vec{v} = v_1^2 + \cdots + v_n^2$  is the square of the vector's length.

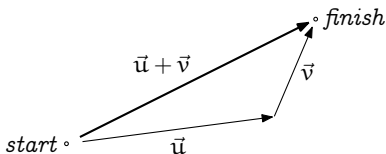
## Triangle Inequality

2.5 *Theorem* For any  $\vec{u}, \vec{v} \in \mathbb{R}^n$ ,

$$|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$$

with equality if and only if one of the vectors is a nonnegative scalar multiple of the other one.

This is the source of the familiar saying, “The shortest distance between two points is in a straight line.”



2.5 *Proof* Since all the numbers are positive, the inequality holds if and only if its square holds.

$$|\vec{u} + \vec{v}|^2 \leq (|\vec{u}| + |\vec{v}|)^2$$

$$(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \leq |\vec{u}|^2 + 2|\vec{u}||\vec{v}| + |\vec{v}|^2$$

$$\vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \leq \vec{u} \cdot \vec{u} + 2|\vec{u}||\vec{v}| + \vec{v} \cdot \vec{v}$$

$$2\vec{u} \cdot \vec{v} \leq 2|\vec{u}||\vec{v}|$$

2.5 *Proof* Since all the numbers are positive, the inequality holds if and only if its square holds.

$$\begin{aligned} |\vec{u} + \vec{v}|^2 &\leq (|\vec{u}| + |\vec{v}|)^2 \\ (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) &\leq |\vec{u}|^2 + 2|\vec{u}||\vec{v}| + |\vec{v}|^2 \\ \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} &\leq \vec{u} \cdot \vec{u} + 2|\vec{u}||\vec{v}| + \vec{v} \cdot \vec{v} \\ 2\vec{u} \cdot \vec{v} &\leq 2|\vec{u}||\vec{v}| \end{aligned}$$

That, in turn, holds if and only if the relationship obtained by multiplying both sides by the nonnegative numbers  $|\vec{u}|$  and  $|\vec{v}|$

$$2(|\vec{v}||\vec{u}|) \cdot (|\vec{u}||\vec{v}|) \leq 2|\vec{u}|^2|\vec{v}|^2$$

and rewriting

$$0 \leq |\vec{u}|^2|\vec{v}|^2 - 2(|\vec{v}||\vec{u}|) \cdot (|\vec{u}||\vec{v}|) + |\vec{u}|^2|\vec{v}|^2$$

is true. But factoring shows that it is true

$$0 \leq (|\vec{u}||\vec{v}| - |\vec{v}||\vec{u}|) \cdot (|\vec{u}||\vec{v}| - |\vec{v}||\vec{u}|)$$

since it only says that the square of the length of the vector  $|\vec{u}||\vec{v}| - |\vec{v}||\vec{u}|$  is not negative.

As for equality, it holds when, and only when,  $|\vec{u}|\vec{v} - |\vec{v}|\vec{u}$  is  $\vec{0}$ . The check that  $|\vec{u}|\vec{v} = |\vec{v}|\vec{u}$  if and only if one vector is a nonnegative real scalar multiple of the other is easy. QED

## Cauchy-Schwarz Inequality

2.6 *Corollary* For any  $\vec{u}, \vec{v} \in \mathbb{R}^n$ ,

$$|\vec{u} \cdot \vec{v}| \leq |\vec{u}| |\vec{v}|$$

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2.6 *Proof* The Triangle Inequality's proof shows that  $\vec{u} \cdot \vec{v} \leq |\vec{u}| |\vec{v}|$  so if  $\vec{u} \cdot \vec{v}$  is positive or zero then we are done. If  $\vec{u} \cdot \vec{v}$  is negative then this holds.

$$|\vec{u} \cdot \vec{v}| = -(\vec{u} \cdot \vec{v}) = (-\vec{u}) \cdot \vec{v} \leq |-\vec{u}| |\vec{v}| = |\vec{u}| |\vec{v}|$$

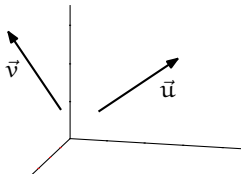
The equality condition is Exercise 19 .

## Angle measure

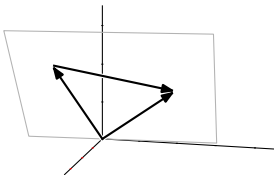
*Definition* The *angle* between two vectors  $\vec{u}, \vec{v} \in \mathbb{R}^n$  is this.

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}\right)$$

We motivate that definition with two vectors in  $\mathbb{R}^3$ .



If neither is a multiple of the other then they determine a plane, because if we put them in canonical position then the origin and the endpoints make three noncolinear points. Consider the triangle formed by  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{u} - \vec{v}$ .



Apply the Law of Cosines:  $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$  where  $\theta$  is the angle that we want to find. The left side gives

$$\begin{aligned} & (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \\ &= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) + (u_3^2 - 2u_3v_3 + v_3^2) \end{aligned}$$

while the right side gives this.

$$(u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - 2|\vec{u}||\vec{v}|\cos\theta$$

Canceling squares  $u_1^2 \dots, v_3^2$  and dividing by 2 gives the formula.

2.8 *Corollary* Vectors from  $\mathbb{R}^n$  are orthogonal, that is, perpendicular, if and only if their dot product is zero. They are parallel if and only if their dot product equals the product of their lengths.