Four.III Laplace's Expansion

Linear Algebra
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Laplace's formula for the determinant

1.1 *Example* Consider the permutation expansion.

$$\begin{vmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{vmatrix} = t_{1,1}t_{2,2}t_{3,3}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{1,1}t_{2,3}t_{3,2}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$
$$+ t_{1,2}t_{2,1}t_{3,3}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{1,2}t_{2,3}t_{3,1}\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$
$$+ t_{1,3}t_{2,1}t_{3,2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + t_{1,3}t_{2,2}t_{3,1}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

Pick a row or column and factor out its entries; here we do the entries in the first row.

$$
\begin{aligned}
&= t_{1,1} \cdot \left[t_{2,2}t_{3,3} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,3}t_{3,2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \right] \\
&\quad + t_{1,2} \cdot \left[t_{2,1}t_{3,3} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,3}t_{3,1} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \right] \\
&\quad + t_{1,3} \cdot \left[t_{2,1}t_{3,2} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + t_{2,2}t_{3,1} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \right]
\end{aligned}
$$

In those permutation matrices, swap to get the first rows into place. This requires one swap to each of the permutation matrices on the second line, and two swaps to each on the third line. (Recall that row swaps change the sign of the determinant.)

$$
\begin{aligned}
&= t_{1,1} \cdot \left[t_{2,2}t_{3,3} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,3}t_{3,2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \right] \\
&\quad - t_{1,2} \cdot \left[t_{2,1}t_{3,3} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,3}t_{3,1} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \right] \\
&\quad + t_{1,3} \cdot \left[t_{2,1}t_{3,2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,2}t_{3,1} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \right]
\end{aligned}
$$

On each line the terms in square brackets involve only the second and third row and column, and simplify to a 2×2 determinant.

$$
= t_{1,1} \cdot \begin{vmatrix} t_{2,2} & t_{2,3} \\ t_{3,2} & t_{3,3} \end{vmatrix} - t_{1,2} \cdot \begin{vmatrix} t_{2,1} & t_{2,3} \\ t_{3,1} & t_{3,3} \end{vmatrix} + t_{1,3} \cdot \begin{vmatrix} t_{2,1} & t_{2,2} \\ t_{3,1} & t_{3,2} \end{vmatrix}
$$

Minor

1.2 *Definition* For any $n \times n$ matrix T, the $(n-1) \times (n-1)$ matrix formed by deleting row i and column j of T is the i, j *minor* of T. The i, j *cofactor* $T_{i,j}$ of T is $(-1)^{i+j}$ times the determinant of the i, j minor of T.

Example For this matrix

$$S = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 4 & -1 \\ 7 & 0 & -3 \end{pmatrix}$$

the $2, 3$ minor is

$$\begin{pmatrix} 3 & 1 \\ 7 & 0 \end{pmatrix}$$

so the associated cofactor is $S_{2,3} = (-1)^5 \cdot (-7) = 7$.

Laplace's formula

1.5 *Theorem* Where T is an $n \times n$ matrix, we can find the determinant by expanding by cofactors on any row i or column j.

$$\begin{aligned} |T| &= t_{i,1} \cdot T_{i,1} + t_{i,2} \cdot T_{i,2} + \cdots + t_{i,n} \cdot T_{i,n} \\ &= t_{1,j} \cdot T_{1,j} + t_{2,j} \cdot T_{2,j} + \cdots + t_{n,j} \cdot T_{n,j} \end{aligned}$$

Proof Exercise 27 . QED

We can find this determinant

$$\begin{vmatrix} 3 & 1 & 2 \\ 5 & 4 & -1 \\ 7 & 0 & -3 \end{vmatrix}$$

by expanding along the second row. Besides $S_{2,3} = 7$, the other two cofactors are here.

$$S_{2,1} = (-1)^3 \cdot \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = 3 \quad S_{2,2} = (-1)^4 \cdot \begin{vmatrix} 3 & 2 \\ 7 & -3 \end{vmatrix} = -23$$

The Laplace expansion gives $5 \cdot 3 + 4 \cdot (-23) - 1 \cdot 7 = -84$.

Adjoint

1.8 *Definition* The matrix *adjoint* (or the *classical adjoint* or *adjugate*) to the square matrix T is

$$\operatorname{adj}(T) = \begin{pmatrix} T_{1,1} & T_{2,1} & \dots & T_{n,1} \\ T_{1,2} & T_{2,2} & \dots & T_{n,2} \\ & \vdots & & \\ T_{1,n} & T_{2,n} & \dots & T_{n,n} \end{pmatrix}$$

where the row i, column j entry, $T_{j,i}$, is the j,i cofactor.

Note that the order of the subscripts in this matrix is opposite to the order that you might expect: the entry above in row i and column j is $T_{j,i}$.

Example The matrix adjoint to this

$$S = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 4 & -1 \\ 7 & 0 & -3 \end{pmatrix}$$

is this (some of these cofactors we have calculated above).

$$\begin{pmatrix} S_{1,1} & S_{2,1} & S_{3,1} \\ S_{1,2} & S_{2,2} & S_{3,2} \\ S_{1,3} & S_{2,3} & S_{3,3} \end{pmatrix} = \begin{pmatrix} -12 & 3 & -9 \\ 8 & -23 & 13 \\ -28 & 7 & 7 \end{pmatrix}$$

1.9 *Theorem* Where T is a square matrix, $T \cdot \mathrm{adj}(T) = \mathrm{adj}(T) \cdot T = |T| \cdot I$. Thus if T has an inverse, if $|T| \neq 0$, then $T^{-1} = (1/|T|) \cdot \mathrm{adj}(T)$.

This summarizes.

$$\begin{pmatrix} t_{1,1} & t_{1,2} & \ldots & t_{1,n} \\ t_{2,1} & t_{2,2} & \ldots & t_{2,n} \\ & \vdots & & \\ t_{n,1} & t_{n,2} & \ldots & t_{n,n} \end{pmatrix} \begin{pmatrix} T_{1,1} & T_{2,1} & \ldots & T_{n,1} \\ T_{1,2} & T_{2,2} & \ldots & T_{n,2} \\ & \vdots & & \\ T_{1,n} & T_{2,n} & \ldots & T_{n,n} \end{pmatrix}$$
$$= \begin{pmatrix} |T| & 0 & \ldots & 0 \\ 0 & |T| & \ldots & 0 \\ & \vdots & & \\ 0 & 0 & \ldots & |T| \end{pmatrix}$$

1.9 *Proof* Theorem 1.5 says we can calculate the determinant of an $n \times n$ matrix T by taking linear combinations of entries from a row and their associated cofactors.

$$t_{i,1} \cdot T_{i,1} + t_{i,2} \cdot T_{i,2} + \cdots + t_{i,n} \cdot T_{i,n} = |T|$$

This immediately gives the diagonal entries of the matrix result of $T\,\mathrm{adj}(T)$.

For the off-diagonal entries, recall that a matrix with two identical rows has a determinant of 0. Thus, for any matrix T, weighing the cofactors by entries from row k with $k \neq i$ gives 0

$$t_{i,1} \cdot T_{k,1} + t_{i,2} \cdot T_{k,2} + \cdots + t_{i,n} \cdot T_{k,n} = 0$$

because it represents the expansion along the row k of a matrix with row i equal to row k. QED

Example The inverse of this matrix

$$S = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 4 & -1 \\ 7 & 0 & -3 \end{pmatrix}$$

is this.

$$\frac{1}{|S|} \cdot \operatorname{adj}(S) = \frac{1}{-84} \cdot \begin{pmatrix} -12 & 3 & -9 \\ 8 & -23 & 13 \\ -28 & 7 & 7 \end{pmatrix}$$

Note The formulas from this section are useful for theory, and for computations with small or special-case matrices. But they are not the best choice for computations with arbitrary matrices because they use more arithmetic than the Gauss-Jordan method.