

One.II Linear Geometry

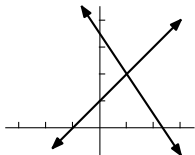
Linear Algebra edition four
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<http://joshua.smcvt.edu/linearalgebra>

Geometry

We can draw two-unknown equations as lines. Then the three possibilities for solution sets become clear.

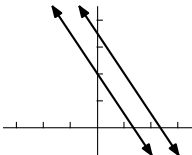
Unique solution



$$3x + 2y = 7$$

$$x - y = -1$$

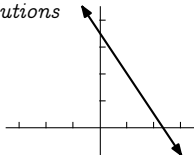
No solutions



$$3x + 2y = 7$$

$$3x + 2y = 4$$

Infinitely many solutions



$$3x + 2y = 7$$

$$6x + 4y = 14$$

This is a nice restatement of the possibilities; the geometry gives us insight into what can happen with linear systems.

Vectors in space

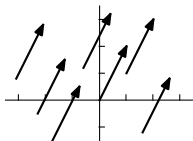
Vectors

A *vector* is an object consisting of a magnitude and a direction.



A vector can model a displacement.

Two vectors with the same magnitude and same direction, such as all of these, are equal.



For instance, each of the above could model a displacement of one over and two up.

Denote the vector that extends from (a_1, a_2) to (b_1, b_2) by

$$\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

so the “one over, two up” vector would be written in this way.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

We often picture a vector

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

as starting at the origin. From there \vec{v} extends to (v_1, v_2) and we may refer to it as “the point \vec{v} ” so that we may call each of these \mathbb{R}^2 .

$$\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\} \quad \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$$

These definitions extend to higher dimensions. The vector that starts at (a_1, \dots, a_n) and ends at (b_1, \dots, b_n) is represented by this column

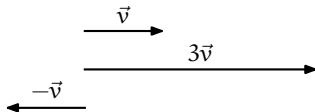
$$\begin{pmatrix} b_1 - a_1 \\ \vdots \\ b_n - a_n \end{pmatrix}$$

and two vectors are equal if they have the same representation. Also, we aren't too careful about distinguishing between a point and the vector which, when it starts at the origin, ends at that point.

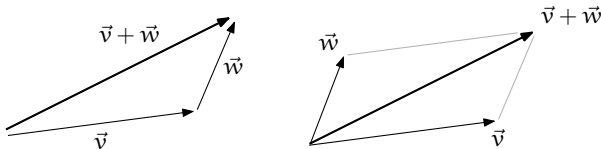
$$\mathbb{R}^n = \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mid v_1, \dots, v_n \in \mathbb{R} \right\}$$

Vector operations

Scalar multiplication makes a vector longer or shorter, including possibly flipping it around.



Where \vec{v} and \vec{w} represent displacements, the vector sum $\vec{v} + \vec{w}$ represents those displacements combined.

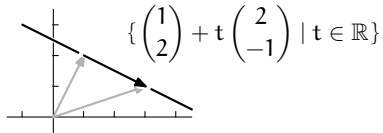


The second drawing shows the *parallelogram rule* for vector addition.

Lines

The line in \mathbb{R}^2 through $(1, 2)$ and $(3, 1)$ is comprised of the vectors in this set

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$



(that is, it is comprised of the endpoints of those vectors). The vector associated with the parameter t

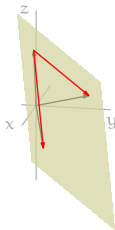
$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

is a *direction vector* for the line. Lines in higher dimensions work the same way.

Planes

The plane in \mathbb{R}^3 through the points $(1, 0, 5)$, $(2, 1, -3)$, and $(-2, 4, 0.5)$ consists of (endpoints of) the vectors in this set.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -8 \end{pmatrix} + s \begin{pmatrix} -3 \\ 4 \\ -4.5 \end{pmatrix} \mid t, s \in \mathbb{R} \right\}$$



The column vectors associated with a parameter, shown here in red,

$$\begin{pmatrix} 1 \\ 1 \\ -8 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} \quad \begin{pmatrix} -3 \\ 4 \\ -4.5 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0.5 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

have their whole body in the plane, not just their endpoint.

A set of the form $\{\vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k \mid t_1, \dots, t_k \in \mathbb{R}\}$ where $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ and $k \leq n$ is a *k-dimensional linear surface* (or *k-flat*).

Length and angle measures

Length

2.1 *Definition* The *length* of a vector $\vec{v} \in \mathbb{R}^n$ is the square root of the sum of the squares of its components.

$$|\vec{v}| = \sqrt{v_1^2 + \cdots + v_n^2}$$

Example The length of

$$\vec{v} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$$

is $|\vec{v}| = \sqrt{1 + 4 + 9} = \sqrt{14}$.

For any nonzero vector \vec{v} , the length one vector with the same direction is $\vec{v}/|\vec{v}|$. We say that this *normalizes* \vec{v} to unit length.

$$\frac{\vec{v}}{|\vec{v}|} = \begin{pmatrix} -1/\sqrt{14} \\ -2/\sqrt{14} \\ -3/\sqrt{14} \end{pmatrix}$$

Note that $\sqrt{(-1/\sqrt{14})^2 + (-2/\sqrt{14})^2 + (-3/\sqrt{14})^2}$ equals 1.

Dot product

2.3 *Definition* The *dot product* (or *inner product* or *scalar product*) of two n -component real vectors is the linear combination of their components.

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Example The dot product of two vectors

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} = 3 - 3 - 4 = -4$$

is a scalar, not a vector.

The dot product of a vector with itself $\vec{v} \cdot \vec{v} = v_1^2 + \cdots + v_n^2$ is the square of the vector's length.

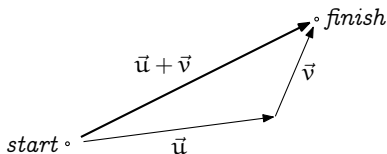
Triangle Inequality

2.5 *Theorem* For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$$

with equality if and only if one of the vectors is a nonnegative scalar multiple of the other one.

This is the source of the familiar saying, “The shortest distance between two points is in a straight line.”



2.5 *Proof* Since all the numbers are positive, the inequality holds if and only if its square holds.

$$\begin{aligned} |\vec{u} + \vec{v}|^2 &\leq (|\vec{u}| + |\vec{v}|)^2 \\ (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) &\leq |\vec{u}|^2 + 2|\vec{u}||\vec{v}| + |\vec{v}|^2 \\ \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} &\leq \vec{u} \cdot \vec{u} + 2|\vec{u}||\vec{v}| + \vec{v} \cdot \vec{v} \\ 2\vec{u} \cdot \vec{v} &\leq 2|\vec{u}||\vec{v}| \end{aligned}$$

That, in turn, holds if and only if the relationship obtained by multiplying both sides by the nonnegative numbers $|\vec{u}|$ and $|\vec{v}|$

$$2(|\vec{v}||\vec{u}|) \cdot (|\vec{u}||\vec{v}|) \leq 2|\vec{u}|^2|\vec{v}|^2$$

and rewriting

$$0 \leq |\vec{u}|^2|\vec{v}|^2 - 2(|\vec{v}||\vec{u}|) \cdot (|\vec{u}||\vec{v}|) + |\vec{u}|^2|\vec{v}|^2$$

is true. But factoring shows that it is true

$$0 \leq (|\vec{u}||\vec{v}| - |\vec{v}||\vec{u}|) \cdot (|\vec{u}||\vec{v}| - |\vec{v}||\vec{u}|)$$

since it only says that the square of the length of the vector $|\vec{u}||\vec{v}| - |\vec{v}||\vec{u}|$ is not negative.

As for equality, it holds when, and only when, $|\vec{u}|\vec{v} - |\vec{v}|\vec{u}$ is $\vec{0}$. The check that $|\vec{u}|\vec{v} = |\vec{v}|\vec{u}$ if and only if one vector is a nonnegative real scalar multiple of the other is easy. QED

Cauchy-Schwarz Inequality

2.6 *Corollary* For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\vec{u} \cdot \vec{v}| \leq |\vec{u}| |\vec{v}|$$

with equality if and only if one vector is a scalar multiple of the other.

2.6 *Proof* The Triangle Inequality's proof shows that $\vec{u} \cdot \vec{v} \leq |\vec{u}| |\vec{v}|$ so if $\vec{u} \cdot \vec{v}$ is positive or zero then we are done. If $\vec{u} \cdot \vec{v}$ is negative then this holds.

$$|\vec{u} \cdot \vec{v}| = -(\vec{u} \cdot \vec{v}) = (-\vec{u}) \cdot \vec{v} \leq |-\vec{u}| |\vec{v}| = |\vec{u}| |\vec{v}|$$

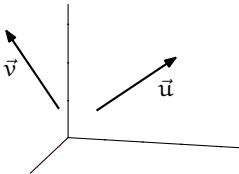
The equality condition is Exercise 19 .

Angle measure

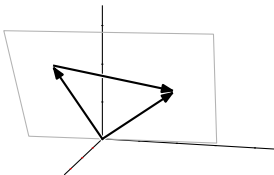
Definition The *angle* between two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is this.

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}\right)$$

We motivate that definition with two vectors in \mathbb{R}^3 .



If neither is a multiple of the other then they determine a plane, because if we put them in canonical position then the origin and the endpoints make three noncolinear points. Consider the triangle formed by \vec{u} , \vec{v} , and $\vec{u} - \vec{v}$.



Apply the Law of Cosines: $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$ where θ is the angle that we want to find. The left side gives

$$\begin{aligned} & (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \\ &= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) + (u_3^2 - 2u_3v_3 + v_3^2) \end{aligned}$$

while the right side gives this.

$$(u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - 2|\vec{u}||\vec{v}|\cos\theta$$

Canceling squares $u_1^2 \dots, v_3^2$ and dividing by 2 gives the formula.

2.8 *Corollary* Vectors from \mathbb{R}^n are orthogonal, that is, perpendicular, if and only if their dot product is zero. They are parallel if and only if their dot product equals the product of their lengths.