# Five.I Complex Vector Spaces

*Linear Algebra*

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# Chapter Five. Similarity

## Scalars will now be complex

This chapter requires that we factor polynomials. But many polynomials do not factor over the real numbers; for instance, $x^2 + 1$ does not factor into a product of two linear polynomials with real coefficients; instead it requires complex numbers $x^2 + 1 = (x - i)(x + i)$.

Consequently in this chapter we shall use complex numbers for our scalars, including entries in vectors and matrices. That is, we shift from studying vector spaces over the real numbers to vector spaces over the complex numbers. Any real number is a complex number and in this chapter most of the examples use only real numbers but nonetheless, the critical theorems require that the scalars be complex.

# Review of Factoring and Complex Numbers

# Division Theorem for Polynomials

Consider a polynomial $p(x) = c_n x^n + \cdots + c_1 x + c_0$ with leading coefficient $c_n \neq 0$. The degree of the polynomial is $n$. If $n = 0$ then $p$ is a constant polynomial $p(x) = c_0$. Constant polynomials that are not the zero polynomial, $c_0 \neq 0$, have degree zero. We define the zero polynomial to have degree $-\infty$.

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So, $x^2 + x$ goes $3x - 1$ times into $3x^3 + 2x^2 - x + 4$ with remainder 4. In $n = dq + r$ form: $3x^3 + 2x^2 - x + 4 = (x^2 + x) \cdot (3x - 1) + 4$.

1.2 *Theorem* Let $p(x)$ be a polynomial. If $d(x)$ is a non-zero polynomial then there are *quotient* and *remainder* polynomials $q(x)$ and $r(x)$ such that

$$p(x) = d(x) \cdot q(x) + r(x)$$

where the degree of $r(x)$ is strictly less than the degree of $d(x)$.

1.4 *Corollary* The remainder when $p(x)$ is divided by $x - \lambda$ is the constant polynomial $r(x) = p(\lambda)$.

*Proof* The remainder must be a constant polynomial because it is of degree less than the divisor $x - \lambda$. To determine the constant, take the theorem's divisor $d(x)$ to be $x - \lambda$ and substitute $\lambda$ for $x$. QED

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If a divisor $d(x)$ goes into a dividend $p(x)$ evenly, meaning that $r(x)$ is the zero polynomial, then $d(x)$ is a called a factor of $p(x)$. Any root of the factor, any $\lambda \in \mathbb{R}$ such that $d(\lambda) = 0$, is a root of $p(x)$ since $p(\lambda) = d(\lambda) \cdot q(\lambda) = 0$.

1.5 *Corollary* If $\lambda$ is a root of the polynomial $p(x)$ then $x - \lambda$ divides $p(x)$ evenly, that is, $x - \lambda$ is a factor of $p(x)$.

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*Proof* By the above corollary $p(x) = (x - \lambda) \cdot q(x) + p(\lambda)$. Since $\lambda$ is a root, $p(\lambda) = 0$ so $x - \lambda$ is a factor. QED

## Factoring over the real numbers

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1.7 *Corollary* Any polynomial with real coefficients factors into a product of linear and irreducible quadratic polynomials with real coefficients. That factorization is unique; any two factorizations have the same factors raised to the same powers.

## Factoring over the complex numbers

While $x^2 + 1$ has no real roots and so doesn't factor over the real numbers, if we imagine a root — traditionally denoted $i$, so that $i^2 + 1 = 0$ — then $x^2 + 1$ factors into a product of linears $(x - i)(x + i)$. When we adjoin this root $i$ to the reals and close the new system with respect to addition and multiplication then we have the complex numbers $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}$. (These are often pictured on a plane with $a$ plotted on the horizontal axis and $b$ on the vertical; note that the distance of the point from the origin is $|a + bi| = \sqrt{a^2 + b^2}$.)

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1.11 *Theorem* [Fundamental Theorem of Algebra] Polynomials with complex coefficients factor into linear polynomials with complex coefficients. The factorization is unique.

# Complex Representations

Recall the definitions of the complex number addition

$$(a+bi)\;+\;(c+di)=(a+c)+(b+d)i$$

and multiplication.

$$\begin{aligned}(a+bi)(c+di)&=ac+adi+bci+bd(-1)\\&=(ac-bd)+(ad+bc)i\end{aligned}$$

With those rules for scalars, all of the operations that we've covered for real vector spaces carry over unchanged.

*Example*

$$\begin{pmatrix}2-i & 1+i\\ i & 4\end{pmatrix}\begin{pmatrix}0 & 3+3i\\ 1-i & 2\end{pmatrix}=\begin{pmatrix}2 & 9+3i\\ 4-4i & 5+3i\end{pmatrix}$$

We shall carry over unchanged from the previous work everything else that we can. For instance, this

$$\left\langle \begin{pmatrix} 1+0i \\ 0+0i \\ \vdots \\ 0+0i \end{pmatrix}, \dots, \begin{pmatrix} 0+0i \\ 0+0i \\ \vdots \\ 1+0i \end{pmatrix} \right\rangle$$

is the *standard basis* for $\mathbb{C}^n$ as a vector space over $\mathbb{C}$ and we denote it $\mathcal{E}_n$.